ARO 16403.1-M

LEVEL II

AD A090842

LANCHESTER-TYPE MODELS OF WARFARE

VOLUME I.

by

James G. Taylor

Professor of Operations Research

Naval Postgraduate School
Monterey, California

80 10 20 019

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 16403.1-M	2. GOVT ACCESSION NO. AD-A090842	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Lanchester-Type Models of Warfare, VOLUME I		5. TYPE OF REPORT & PERIOD COVERED Final Report 1 Oct 79 — 30 Sep 80
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) James G. Taylor		8. CONTRACT OR GRANT NUMBER(s) ARO MIPR 34-79
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School, Monterey, CA 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS MIPR-ARO-34-79
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, Post Office Box 12211, Research Triangle Park, NC 27709		12. REPORT DATE Oct 80
		13. NUMBER OF PAGES 606
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

monographs	Lanchester type models
warfare	differential equations
operations research	attrition
combat operations	mathematical models

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This monograph is a comprehensive treatise on Lanchester-type models of warfare, i.e. differential-equation models of attrition in force-on-force combat operations. Its goal is to provide both an introduction to and current-state-of-the-art overview of Lanchester-type models of warfare as well as a comprehensive and unified in-depth treatment of them. Both deterministic as well as stochastic models are considered. Such models have been widely used in the United States and elsewhere for the modelling of force-on-force attrition over the complete spectrum of combat operations, from

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT CONTINUED

combat between platoon-sized units through theater-level air-ground combat. This material should be of interest primarily to individuals concerned with defense planning, quantitative aspects of military analysis, military OR, war gaming, or combat modelling, although it may also be of interest to the reader concerned with the modelling and analysis of other dynamic systems. It should also be of interest to the concerned citizen who is interested in the foundations for defense analysis and has the appropriate technical background.

PREFACE

The Twentieth Century has been characterized by innumerable attempts to use the Scientific Method as a basis for policy planning in national and international affairs. The emergence of the field of operations research (OR) out of attempts of scientists in the Western Democracies to apply the Scientific Method to military problems during World War II is well known. Since World War II there has been a dramatic growth in both the interest in and use of OR and systems-analysis techniques for such purposes within the U.S. defense establishment, especially since the beginning of the so-called McNamara Era of defense planning. A concomitant trend has been an equally dramatic increase in both the number and variety of mathematical models used to support these analytical activities.

Unfortunately, professional communications within the defense analytical community have not kept pace with this dramatic growth in modelling and analysis activities. In particular, there has been a relative lack of scientific communication and organization of knowledge concerning the foundations of defense analyses and associated defense-analysis technology. However, even this important point has not been explicitly articulated in several fairly recent critical appraisals of the foundations of defense analyses[†]. To be sure, research progress on these foundations has been made, but it has not always been efficiently and effectively communicated to interested parties. This inaccessibility

[†] In particular, see JACOB A. STOCKFISCH, "Models, Data, and War: A Critique of the Study of Conventional Forces," R-1526-PR, The RAND Corporation, Santa Monica, California, March 1975 and also U.S. General Accounting Office, "Models, Data, and War: A Critique of the Foundation for Defense Analyses," PAD-80-21, Washington, D.C., March 1980.

of scientific information concerning combat-modelling methodologies has contributed to the existing gap between theory and practice. Some undesirable consequences of this communications deficiency between analysts and researchers include (1) duplication of effort, (2) models being inefficiently used (or even misused), (3) lack of the appropriate intellectual environment for effective professional review by peers, and (4) lack of any "road map" to provide direction (and purpose) for methodological developments.

Thus, although there has been a great need, information about combat-modelling methodologies, their strengths and weaknesses, limitations, etc. has not been very widely disseminated in accessible form. National security (i.e. material being classified) has not really been a factor in producing this situation in which the quantitative foundations of defense analyses have not been readily available to the analysis community for scientific scrutiny. Without such generally available methodological material, little scientific progress can be made, since open scientific discussion is hampered by such vital information not being readily available to all interested parties. Consequently, this monograph has been written in an attempt to fill some of this void by organizing the current state of knowledge about a certain type of combat model, so-called LANCHESTER-type equations of warfare. Hopefully, its appearance will also stimulate discussion and debate concerning assessment of existing capabilities and future needs in this one specific area of combat-modelling methodology.

At the personal level, the reader may be interested in knowing how the author has become drawn to this subject: the author has been interested in the subject of LANCHESTER-type combat models since the late

1960's, when R. NICHOLS HAZELWOOD introduced him to combat models and, in particular, to the work of HERBERT K. WEISS. He has been fortunate enough to have subsequently had such interests nurtured at the Naval Postgraduate School (NPS) and has had the opportunity to do research on combat models and teach graduate-level courses about them to students (primarily U.S. Army and U.S. Marine Corps officers) in the OR curriculum at NPS since 1970. The treatise at hand (and its petite predecessor Force-on-Force Attrition Modelling[††]) has evolved from these activities.

This monograph is a comprehensive treatise on LANCHESTER-type models of warfare, i.e. differential-equation models of attrition in force-on-force combat operations. Its goal is to provide both an introduction to and current-state-of-the-art overview of LANCHESTER-type models of warfare as well as a comprehensive and unified in-depth treatment of them. Both deterministic as well as stochastic models are considered. Such models have been widely used in the United States and elsewhere for the modelling of force-on-force attrition over the complete spectrum of combat operations, from combat between platoon-sized units through theater-level air-ground combat. This material should be of interest primarily to individuals concerned with defense planning, quantitative aspects of military analysis, military OR, war gaming, or combat modelling, although it may also be of interest to the reader concerned with the modelling and analysis of other dynamic systems. It should also be of interest to the concerned citizen who is interested in the foundations for defense analysis and has the appropriate technical background.

[††] The full citation here is JAMES G. TAYLOR, Force-on-Force Attrition Modelling, Military Applications Section of the Operations Research Society of America, Arlington, Virginia, 1980.

I have tried to make this monograph particularly suitable for three specific groups of readers: (1) the beginning student of military OR, (2) the practicing military OR analyst, and (3) the research worker in OR, applied mathematics, models, or systems analysis and evaluation. For the first group (i.e. beginning students of military OR), I have included much expository and explanatory material: each major topic is preceded by a general discussion of the contextual setting in which it arises (with figures depicting important conceptual ideas and typical numerical results). For these readers I have supplied motivation and overview. For the second group (i.e. practicing military OR analysts), I have emphasized those theoretical and applied concepts that are basic for the building and running of operational combat models (e.g. the numerical determination of values for LANCHESTER attrition-rate coefficients) and have provided a bridge between such current operational combat models and the abstract notions that form their conceptual bases. For these readers I have supplied examples from current operational combat models. For the third group (i.e. OR and other researchers), I have surveyed the current state of the art of pertinent quantitative methodologies concerning LANCHESTER-type combat models, particularly mathematical results for analytically investigating the quantitative behavior of relatively simple LANCHESTER-type models. For these readers I have included numerous references to the literature and a comprehensive bibliography on the LANCHESTER theory of combat. This book, however, is particularly slanted toward the beginning military-OR student who is interested in force-on-force combat models, since it is through him (particularly if he is an officer in one of the military services) and his education about combat models that the greatest long-term improvements in defense decision

making may be achieved by the U.S. Department of Defense (DoD). It strives to give the reader (regardless of his orientation) an appreciation of the complex operational models that are today used for investigating large-scale simulated air-ground combat operations by DoD. Mathematical prerequisites have been kept to a minimum, with more mathematically oriented sections that are not necessary for the understanding of the sequel being identified as "starred sections." Throughout this monograph, modelling aspects have been emphasized. Anyone with a background in calculus good enough to understand the physical interpretation of an ordinary-differential equation model should have no trouble in reading most of it. However, the few starred sections do require more mathematical sophistication to be understood.

This monograph is organized into two volumes of four chapters each. The monograph begins with a discussion in Chapter 1 about the general nature of models (particularly, combat models), their use in OR, and particularly the contextual setting for the use of such models as planning tools in the U.S. DoD. Chapter 2, which begins by reviewing FREDERICK W. LANCHESTER's pioneering work on quantitatively justifying the Principle of Concentration, examines LANCHESTER's classic combat models and the many subsequent variants of them. The models are kept simple and deterministic here, but the stage is set for subsequent model enrichments considered later in this monograph. The discussion of LANCHESTER's classic combat models is self-contained, with background material on the relevant mathematics being contained in an appendix. This material is fundamental and very important not only in its own right but also for understanding subsequent developments in this book: it forms the basis for the many extensions considered later in the book. A

selection of problems has been provided in Chapter 2 for the enhancement of the reader's familiarity with these basic models.

Chapter 3 contains a comprehensive examination of some simple models of battle termination. It considers both the empirical foundations of such models and also the mathematical analysis of their properties. Both deterministic and stochastic battle-termination processes are examined, although only deterministic LANCHESTER-type attrition processes are considered. This chapter is essentially a state-of-the-art survey of battle-termination modelling and focuses on work by H.K. WEISS and R.L. HELMBOLD. It culminates by examining HELMBOLD's empirical investigation of the validity of breakpoint hypotheses. Chapter 4 examines stochastic versions of the simple deterministic homogeneous-force models considered in Chapter 2. Continuous-time MARKOV-chain models of LANCHESTER-type attrition processes are exclusively considered. After examining analytical results for such models and noting their complexity, the reader will certainly appreciate the fact that except for small numbers of combatants, the expected course of combat (at least for MARKOV-chain models of homogeneous-force combat) is well approximated by deterministic LANCHESTER-type equations. Not surprisingly, such deterministic LANCHESTER-type models are consequently frequently referred to as expected-value models. Herein ends Volume I.

Volume II begins with Chapter 5. In order to use a LANCHESTER-type model in any actual military OR study, numerical values must be determined for the attrition-rate coefficients, which represent the single weapon-system-type kill rates. Chapter 5 considers in detail approaches and methodologies for determining such numerical values for LANCHESTER attrition-rate coefficients for various types of weapon systems. The

two main approaches that are currently used in the United States to determine such single-system kill rates are based on using (1) a "free-standing" analytical submodel of an individual firer engaging a single enemy target, and (2) a statistical estimate based on "combat" data generated by a detailed Monte Carlo combat simulation. Such methodology is a basic essential ingredient for the building of any operational LANCHESTER-type combat model. Chapter 6 considers LANCHESTER-type models for combat between two homogeneous forces and emphasizes the analysis of such models. For several important classes of homogeneous-force models, analytical results are given that make the analysis (including determining the force levels as functions of time and predicting the battle's outcome) of such variable-coefficient combat models almost as convenient as that of LANCHESTER's original constant-coefficient ones. Tables of special new mathematical functions (i.e. the LCS functions developed by the author) are provided for the reader's use in analyzing certain important classes of "aimed-fire" battles between two homogeneous forces.

Chapter 7 considers modelling tactical engagements and surveys approaches currently used in the United States for assessing casualties in simulated tactical engagements between general-purpose military forces in conventional air-ground combat operations. It reviews the various different modelling alternatives available to the military OR worker and then expounds on both detailed deterministic LANCHESTER-type models of attrition in tactical engagements and also aggregated-force models based on index numbers (e.g. firepower scores), with hierarchical modelling approaches also being briefly discussed. Model formulation and methodological aspects are emphasized, with simple auxiliary models

being used to illustrate modelling points for developing and understanding complex operational models. Examples of current operational models that use the two main theoretical approaches of casualty assessment (i.e. detailed LANCHESTER-type force-change representations and aggregated-force casualty assessments based on index numbers) are given. Recent developments by authors such as L.B. ANDERSON, D.P. DARE, and R.M. THRALL for determining firepower scores (i.e. weapon-system-type values) from a linear model that imputes values to weapon-system types based on their LANCHESTER attrition-rate coefficients are reviewed and discussed, as well as the important (and elusive) problem of historical validation of attrition models. Next, Chapter 8 reviews work on developing insights into the structure of optimal tactical decisions by applying the appropriate optimization theory to a combat model with military strategy and tactics quantified through tactical-choice variables. Gaming aspects are also briefly considered. This chapter is essentially a comprehensive overview and review of work on the quantitative study of military strategy and tactics by using optimization theory in conjunction with combat-modelling theory. Again, simple auxiliary LANCHESTER-type models are used to study these complex operational problems. As before, model formulation and insights gained into the structure of optimal time-sequential decisions are stressed, with optimization-theory (i.e. differential-game) prerequisites being kept at a minimum (i.e. the results of such optimization studies are given but not the details in the application of the optimization theory). Finally, a comprehensive bibliography on the LANCHESTER theory of combat is included in an appendix for the reader who is interested in further information about it.

This monograph has evolved out of a tutorial on LANCHESTER-type

models of warfare that the author was invited to deliver by the Military Applications Section of the Operations Research Society of America (ORSA) at the 46th National ORSA Meeting on Thursday October 17, 1974 in San Juan, Puerto Rico. This tutorial was well received, and it was subsequently repeated at the 35th Military Operations Research Symposium in July 1975 and at the 15th Annual U.S. Army Operations Research Symposium in October 1976. After attending this tutorial in July 1975, CDR JAMES J. MARTIN, USN, then Chairman of the MORS Publications Committee, expressed strong interest in the author's expanding the tutorial material into a monograph on LANCHESTER-type models of warfare. The writing of this monograph was consequently begun under the sponsorship of the Office of Naval Research (Code 431, Naval Analysis Programs) in July 1976. Continued encouragement by Dr. MARTIN (now retired from the U.S. Navy) has been appreciated. I have used earlier drafts of the beginning portions of this material (primarily Chapters 1 and 2 and occasionally Chapter 3) in graduate courses on combat models for OR students at the Naval Postgraduate School.

The author would like to thank all the organizations and individuals who have helped facilitate the appearance of this monograph. Although all those who have helped me are far too numerous to mention, I would like to explicitly express my thanks to several. In particular, the writing of this monograph has been financially supported by the Office of Naval Research (both through direct funding by Code 431 and also through the Foundation Research Program at the Naval Postgraduate School), the U.S. Army Research Office (ARO), Durham, North Carolina, and the Headquarters of the USAF, Studies and Analysis Group. Additionally, ARO supported some separate research during this period on

LANCHESTER-type models of warfare, and results from this work have been incorporated into the monograph at hand. Most of the author's research on LANCHESTER-type models of warfare, however, has been supported over a number of years by the Office of Naval Research (both through direct funding by Code 431 and also through the Foundation Research Program at NPS). The author would like to thank Provost JACK R. BORSTING of NPS (formerly chairman of the OR department) for his continual encouragement and support of such work as well as that from subsequent OR department chairmen Dean DAVID A. SCHRADY and Professor MICHAEL G. SOVEREIGN. The endeavors of Associate Professor GILBERT T. HOWARD (associate chairman for research of the OR department) in this respect are also gratefully acknowledged. The author would also like to thank HERBERT K. WEISS, Dr. JAMES J. MARTIN, Dr. FRANK E. GRUBBS, Professor MARTIN SHUBIK, and LTC JOHN FRIEL (USAF), for their constant encouragement. Additionally, the author would like to thank Professors CLINTON J. ANCKER, GORDON E. LATTA, GUILLERMO OWEN, and MICHAEL G. SOVEREIGN, as well as LTC RICHARD S. MILLER (USA) for their numerous suggestions for improving this manuscript. I am especially indebted to LTC MILLER for many stimulating discussions on the topics of combat modelling and this constant encouragement and help concerning this project. The author would also like to thank the late ROSEMARIE STAMPFEL[†††] for her consummate typing of this manuscript. Finally, the author would like to thank his family for their understanding of the long hours he has spent

---

[†††] Sadly and unexpectedly ROSEMARIE STAMPFEL passed away just after completing the typing of the first draft of the manuscript. As a technical typist, she was without peer. I would like to thank her for her many suggestions and help in improving this manuscript. She will be missed by many.

writing this book and for their constant support, especially his wife MARY ANN, who has proofread most of this monograph (some while recovering from surgery).

TABLE OF CONTENTS

VOLUME I.

Chapter 1. Background and Introduction 1

1.1 Operations Research and Models 1
1.1.1 The General Nature of Operations Research 1
1.1.2 The General Nature of Models 3
1.2 Defense Planning, Combat Models, and the Scientific Study of Warfare 6
1.3 Different Types of Combat Models 10
1.4 The Influence of Modern Digital Computer Technology.................. 27
1.5 The Purpose of This Book 27
1.6 Dynamic Systems and State Variables 30
1.7 Final Remarks 31

Footnotes 33
Notes and Remarks 40
References 44

Chapter 2. Lanchester's Classic Combat Formulations 52

2.1 Lanchester's Original Work 52
2.2 Constant-Coefficient Lanchester-Type Equations for Modern Warfare.. 63
*2.3 A Further Look at Force Annihilation 84
2.4 Constant-Coefficient Lanchester-Type Equations for Area Fire 91
*2.5 A Further Look at the Area-Fire Model 103
2.6 Shortcomings of Lanchester's Original Models 110
2.7 Subsequent Development of the Lanchester Theory of Combat: A Preview of Things to Come 115
2.8 A Simple Model of Battle Termination 123
2.9 Concentration of Forces Revisited 141
*2.10 Fiske-Type Equations of Warfare 155
2.11 Comparison of Lanchester's Two Basic Models and Summary 159
2.12 A Classification Scheme for Homogeneous-Force Lanchester-Type Attrition Processes and Some Additional Functional Forms for Attrition Rates 167

Problems 182
Footnotes 203
References 215

Appendix A. Background for the Mathematics of Lanchester's Classic Combat Formulations 222
A.1 The Hyperbolic Functions 223
A.2 Solution to the $n^{\underline{th}}$ Order Constant-Coefficient Linear Differential Equation 228
A.3 The Generalized Riccati Equation 232

*Starred sections are not required for the understanding of the sequel and should be omitted at first meeting.

Chapter 3. Some Simple Models of Battle Termination 234

3.1 Introduction 234
3.2 Modelling Battle Termination 236
3.3 Development Battle-Outcome-Prediction Conditions 241
3.4 Modelling a Unit's Force-Level Breakpoint 254
3.5 Victory-Prediction Conditions for Deterministic Lanchester-Type Attrition Processes with Deterministic Force-Level Breakpoints 259
3.6 Development of Victory-Prediction Conditions for Lanchester's Classic Models with Deterministic Force-Level Breakpoints 262
3.7 Development of General Battle-Outcome-Prediction Conditions for Deterministic Lanchester-Type Attrition Processes with Stochastic Casualty-Fraction Breakpoints 270
3.8 Battle-Outcome-Prediction Conditions for Deterministic FT|FT Attrition Processes with Stochastic Breakpoints 280
3.9 Battle-Outcome-Prediction Conditions for Deterministic F|F Attrition Processes with Stochastic Breakpoints 301
*3.10 Another Model that Considers Unit Deterioration Due to Attrition .. 309
3.11 Weiss's Model of Battle Termination 331
3.12 Weiss's Model of Engagement Outcomes in the U.S. Civil War 336
3.13 Weiss's Empirical Examination of Engagement Outcomes in the U.S. Civil War 343
3.14 Helmbold's Empirical Investigation of the Validity of Breakpoint Hypotheses 356

Footnotes 369
References 378

Appendix B. The Probability that One Random Variable is Less than Another . 380

Chapter 4. Stochastic Lanchester-Type Combat Models 388

4.1 Introduction 388
4.2 Probabilistic Dynamic Models 392
4.3 A Markov-Chain Model for Lanchester-Type Combat 397
4.4 Information to be Obtained from the Model 413
4.5 Verification that P(t,m,n) is a Probability Mass Function 416
4.6 The Distribution of the Times between Casualties for the General Model 420
4.7 The Special Case of Stationary Transition Probabilities 424
4.8 An Important Tie-In to the Modelling of Lanchester Attrition-Rate Coefficients 435
4.9 The State Probabilities 438
4.10 The Probability of Winning 472
4.11 Approximations to the Probability of Winning 486
4.12 The Average Force Levels 493
4.13 Variability in the Mean Course of Combat 508
4.14 Monte Carlo Methods 517
*4.15 Behavior for Large Numbers 526
4.16 Comparison of Deterministic and Stochastic Attrition Models 532

Footnotes 549
References 562

*Appendix C. Solution of the Fundamental Partial-Difference Equation for the Probability that X Wins with m Final Survivors $P_{m,n_{BP}}(m_0,n_0)$ 569

VOLUME II.

Chapter 5. Lanchester Attrition-Rate Coefficients 1

5.1 General Considerations 1
5.2 Attrition-Rate Coefficients for Lanchester's Equations of Modern Warfare 7
5.3 Justification of General Expression for Attrition-rate Coefficients for Lanchester's Equations of Modern Warfare 14
5.4 Bonder's Model for Markov-Dependent Fire 20
5.5 Derivation of Bonder's Result for the Expected Time to Kill a Target (Approach Based on the Exact Distribution of Time to Kill). 27
5.6 A Simple Derivation of the Expected Number of Rounds Necessary to Obtain z Hits 48
5.7 The Number of Rounds Necessary to Kill a Target (General Derivation) 52
5.8 General Results for the Time to Kill a Target 58
5.9 Development of Expected Time to Kill a Target as Mean First-Passage Time in Continuous-Time Semi-Markov Process 72
5.10 Special Cases of Bonder's General Expression for the Lanchester Attrition-Rate Coefficient 81
5.11 Variables Upon Which Attrition-Rate Coefficients Depend 89
5.12 Some Typical Range Dependencies for the Lanchester Attrition-Rate Coefficient 93
5.13 Attrition-Rate Coefficients for Area-Fire Weapons 100
5.14 Results for Other Related Weapon-System Types 117
5.15 Maximum-Likelihood Estimation of Attrition-Rate Coefficients 125
5.16 Attrition-Rate Coefficients for Heterogeneous-Force Combat 148

Footnotes 180
References 196

Chapter 6. Homogeneous-Force Models 203

6.1 Introduction 203
6.2 Bonder's Constant-Speed-Attack Model 206
6.3 Information to be Obtained from the Model 227
6.4 The Special Case of Quasi-Autonomous Equations 230
6.5 General Force-Level Results for Variable-Coefficient Lanchester-Type Equations of Modern Warfare 233
6.6 Force-Annihilation-Prediction Conditions 244
6.7 Parametric Dependence of the Parity-Condition Parameter 262
6.8 Numerically Determining the Parity-Condition Parameter 268
6.9 Application to General Power Attrition-Rate Coefficients 276
6.10 The Liouville-Green-Lanchester Approximation 295
6.11 Helmbold's Modification of Lanchester's Equations 299
6.12 The General Linear Model for Combat Between Two Homogeneous Forces 306
6.13 Combat with Supporting Fires 313
6.14 Helmbold-Type Combat with Supporting Fires 330
6.15 The General Linear Model with Replacements (Constant Attrition-Rate Coefficients) 333

6.16 Variable-Coefficient Equations for FT|FT Attrition Process 341
*6.17 A Result for the General Model with Temporal Variations in Fire Effectiveness 346

Footnotes 355
References 361

Appendix D. Tables of LCS Functions for Analysis of Homogeneous-Force Battles 366

Chapter 7. Modelling Tactical Engagements 426

7.1 Introduction 426
7.2 Additional Operational Factors to be Considered in Lanchester-Type Models 431
7.3 Modelling Small-Scale Engagements versus Modelling Large-Scale Ones 441
7.4 Applications to Guerrilla Warfare 446
7.5 Deitchman's Basic Ambush Model 447
7.6 Schaffer's Models of Guerrilla Engagements 459
7.7 Modelling Attrition for Combat Between Heterogeneous Forces 477
7.8 Analytical Results for Heterogeneous-Force Models 484
7.9 Current Detailed Lanchester-Type Operational Models of Tactical Engagements 498
7.10 Overview of Aggregated-Force Models of Attrition in Tactical Engagements 504
7.11 Aggregation of Forces in Combat Analyses 506
7.12 General Mathematical Structure of Attrition Calculations in Aggregated-Force Models 509
7.13 Fitting a Differential-Equation Model to Loss-Rate Curves Typically Used to Represent Large-Scale Ground-Combat Attrition .. 513
7.14 Changes over Time in the Force Ratio for the Above Model 521
7.15 FEBA-Movement Modelling 525
7.16 Dynamics of FEBA Movement in Large-Scale Ground-Combat Models 531
7.17 Current Complex Aggregated-Force Operational Models of Large-Scale Tactical Engagements 541
*7.18 A Linear Model for Imputing Values to Weapon-System Types Based on Their Lanchester Attrition-Rate Coefficients 542
*7.19 Critique of Such Methodology for Imputing Values to Weapon-System Types 574
7.20 Hierarchical-Modelling Approaches 588
7.21 Significant Modelling Issues 590
7.22 Historical Validation of Attrition Models 593
7.23 The Complexity Crisis 606

Footnotes 609
References 629

Appendix E. Finite-Difference Approximations to Lanchester-Type Equations 642

Chapter 8. Optimizing Tactical Decisions 682

8.1 Introduction 682
8.2 Quantitative Analysis of Military Strategy and Tactics 685
8.3 Information to be Obtained from the Quantitative Analysis of Military Strategy and Tactics 690
8.4 Basic Elements of the Combat-Optimization Problem 693
8.5 Simple Auxiliary Models and Complex Operational Models 697
8.6 Overview of Problems Considered in the Literature 703
8.7 Decision Analysis for Tactical Military Decisions 707
8.8 Some Combat-Optimization Problems to be Briefly Examined Further . 710
8.9 Optimal Initial Commitment of Forces 713
8.10 The Simplest Fire-Distribution Problem 719
8.11 Optimal Control of Lanchester-Type Attrition Processes 738
8.12 Lanchester-Type Differential Games 759
8.13 Insights Gained 770
8.14 Role of Optimization in Decision Analysis for Tactical Military Decisions 772

Footnotes 774
References 784

Appendix F. Comprehensive Bibliography on the Lanchester Theory of Combat 792

Chapter 1. BACKGROUND AND INTRODUCTION

1.1. Operations Research and Models.

Loosely speaking, LANCHESTER-type models of warfare are differential-equation models of combat operations. In one form or another, such models are fairly widely used in operations research (OR) studies by the Department of Defense (DoD) in the United States. The use of these combat models for planning purposes has been made possible by modern large-scale digital-computer technology. However, there are competing methodologies (for example, so-called high-resolution Monte-Carlo simulation) for combat modelling, and there has been much debate[1] by advocates about the advantages of this method or that one for defense planning. To place such discussion about the use (and misuse of combat models, their realm of applicability, and their strengths and weaknesses in proper perspective, it seems appropriate to briefly discuss the nature of OR, combat models, and their use by DoD. The reader should keep in mind, however, that this book will focus on LANCHESTER-type models of warfare.

1.1.1. The General Nature of Operations Research.

Operations research (OR) originated out of questions arising in military activities during World War II. After the war, the approach and techniques of OR were applied to business and non-military government problems. OR has expanded greatly during the thirty or more years since the end of World War II. What exactly is OR? Although there is far from universal agreement[2] as to the exact nature of OR, the author prefers to think of OR in the following terms[3]: operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.

The above definition of OR is not new, but the author feels that it is important because this definition focuses on what is being done and not the techniques used. Moreover, one should expect to find that different methodologies receive different amounts of emphasis in different fields of application of OR. For example, in the private (i.e. business) sector of the economy one finds that the "theory of the firm" and related subjects (such as profit maximization, efficient distribution of products, investment planning, inventory management, etc.) play a central role in OR applications and require the use of certain OR theory and techniques (such as inventory theory, queueing theory, linear and integer programming, discounted cash flow, etc.). One would expect quite a different phenomenological basis for defense planning, with possibly different OR techniques receiving emphasis. It is the author's hypothesis that defense planning should be based as much as possible on the scientific study of warfare. Unfortunately, this is not the case in practice today (see, for example, SHUBIK and BREWER [86, pp. 9-10] for a discussion of this point). For further discussion of the nature of OR, the interested reader should consult the literature[4].

Four concepts of fundamental importance to the practice of OR are (see HERRMANN and MAGEE [38]):

(C1)[5] the model,

(C2) the measure of effectiveness (MOE),

(C3) decision making,

(C4) the role of experimentation.

Models (in particular, so-called LANCHESTER-type models of warfare) are the central theme of this book. We should bear in mind, however, that the development and application of a model in an OR study is only one of several essential ingredients for a successful study. Each of the three other aspects

listed above can significantly contribute to the failure of a defense-planning study. It is the author's opinion that people unfamiliar with quantitative models are quick to blame an unfamiliar modelling methodology for deficiencies in the application (e.g. data-base quality or errors, incorrect implementation, etc.) of a particular model. The practitioner should not blame the model (particularly, a LANCHESTER-type model) if the wrong MOE is used in a study, nor should he blame the modelling methodology if the model is incorrectly applied or exercised with low-quality data, or if the scenario is wrong. Thus, the development of a combat model is only one facet of a military OR study, albeit a very important aspect.

During World War II most OR concerned actual ongoing military operations. Some people prefer to use the term operations analysis (OA) for such activities. In 1976 (with the end of U.S. involvement in Southeast Asia) most applied military OR activities concerned some type of planning. If a military system does not physically exist (and even when it does), its effectiveness must be evaluated "on paper." Thus, for example, for assistance in system-acquisition decisions, one would expect to use in the advanced planning phase some type of combat model to help quantitatively explore the possible benefits from a proposed system. Even if a prototype has been built and "operational" data has been collected, some type of combat model may be required to assess the system's military worth based on the observed performance data.[6] In other words, the nature of military OR has changed since World War II when few operational models were really used, and today combat models are an essential (and expensive[7]) part of DoD planning activities.

1.1.2. The General Nature of Models.

It seems appropriate for us to briefly discuss the general nature of

models in order to better place combat models in proper perspective. Models are basically representations. They may be representations of states, objects, or events. Models are idealizations (i.e. abstractions) in the sense that they are less complicated than reality (and hence potentially easier to use for research purposes). The U.S. Army Models Review Committee [42, Appendix B to Chapter I] has defined a model as "an abstract representation of reality which is used for the purpose of prediction and to develop understanding about the real-world process."

Thus, models are easier to manipulate and "carry about" than the real thing. They are relatively simple compared with reality because only the relevant features of reality have been represented. For the person unacquainted with this basic property of models, however, it is easy to confuse relevance with realism. Thus, many DoD decision makers who are removed from the modelling business find simulations to be more credible models of combat operations than analytical models because of the much larger amount of detail that is present in a simulation. Additionally, models allow one to transcend one's environment and make inferences about things and events that have not been experienced directly. In the analysis of combat operations (particularly possible future ones), this aspect is quite important.

There are many ways to classify models. Three different basic types of models are the following:

(T1) iconic models,

(T2) analogue models,

(T3) symbolic models.

An iconic model is a large- or small-scale representation of states, objects, or events. They "look like" what they are supposed to represent with only

a transformation of scale. Examples of iconic models are a flow chart, blueprint, road-map (or any other type of picture or diagram that looks like the real thing), pilot plant, or a wind tunnel. In each case only the scale of the system or operation has been changed.

An analogue model uses one property to represent another different property. For example, we can represent the third dimension (i.e. elevation) on a two-dimensional map by means of contour lines, which represent information about changes in elevation (i.e. slopes) by their distance apart. Another similar example is the use of colors to represent different types of terrain on a map. Since one property is used to represent another, a legend is required to remind the reader of the transformation of properties. Other examples of analogue models are the slide rule and an electrical system represented by a hydraulic system.

The last general type of model is the symbolic model, which represents properties symbolically. Verbal descriptions of processes or systems qualify as symbolic models When symbols represent quantities, the model is usually called a mathematical model. We will focus on mathematical models of combat (in particular, combat attrition) in this book. Here we have indicated to the reader, however, that other types of models certainly exist.

Although they are the most abstract, the distinguishing feature of mathematical models is the ease with which they may be manipulated for the extraction of information. Iconic and analogue models are much less flexible in this respect. In terms of combat operations, we should point out that field exercises are basically iconic models, while map exercises are basically analogue models. However, both these two types of combat models are difficult to manipulate (particularly the field exercise, which is also very costly). Thus, although they may require some time and cost to develop, mathematical models are relatively easy to manipulate and hence

respond to the demands of analysis.

Many other classifications of models are possible,[8] but for our purpose of studying combat modelling we need only distinguish here between two basic types of mathematical models:

(T1) deterministic model.

and (T2) stochastic model

A <u>deterministic model</u> is one that contains no element of chance. Hence, its output is uniquely determined by its input in the sense that the same input always produces the same output. A <u>stochastic model</u> contains an element of chance (or uncertainty[9]) so that its output is not uniquely determined in this sense by input, but rather one must talk about the chances of observing various outputs for a given input. In other words, one must consider the probability distribution over the set of possible outcomes for a given set of inputs. In this book we will consider both deterministic and stochastic LANCHESTER-type models of warfare.

1.2. <u>Defense Planning, Combat Models, and the Scientific Study of Warfare</u>.

The Twentieth Century has been characterized by attempts to use the Scientific Method in policymaking, in particular for military and defense problems. Many writers[10] have stressed the importance of applying quantitative OR methodologies to defense planning. Enlightened defense planning is, of course, important for both the short-run and also the long-run national security of the United States.[11] What are typical defense-planning problems? According to STOCKFISCH [90], they are as follows:

(P1) How do we assess a possible opponent's military capability, and how large should our military forces be to meet the perceived threat?

(P2) How should the total force be structured between major services, such as land forces and tactical air forces?

(P3) How should the land forces be structured with respect to (1) combat branches, such as infantry and tanks, and (2) service specialties that provide logistic and personnel support?

(P4) What should be the technical performance and physical specifications of new weapons that will be the object of engineering development programs? Given the availability of new weapons, what should be their tactical usage, how many of them should be procured, and in what organizational and command context should they be employed?

Such questions concern the evaluation of weapon-system and force-level planning alternatives in future time frames. In order to determine the benefits to be gained from a particular alternative, one is invariably faced with the problem of predicting the effectiveness of specified military forces in possible future military engagements. Since such forces and/or weapon systems only exist "on paper," some type of combat model (see Section 1.3 for further details) must be used in such studies. In way of summary, then, combat models are valuable in many aspects of defense planning: (1) for evaluating "on paper" proposed weapon systems during advanced planning; (2) for extending, interpolating, and interpreting operational test data during field testing; etc. (see [104] for a fuller discussion).

Thus, combat models have been used as decision aids for defense planning. They have actually been used by analysts to study such major subjects (see STOCKFISCH [90]) as:

(S1) the design specification and selection of new weapons,

(S2) the allocation of resources between air and land forces and, within land forces, between infantry and artillery,

(S3) how tactical air capability might be allocated among diverse missions,

(S4) the amount of logistic support that the combat elements of field forces should have,

(S5) the rate at which forces might be mobilized and deployed,

and (S6) the issue of how large the forces should be.

The kinds of models that are used for such studies should be related to the type of information that is desired from the analysis. We will discuss the various types of combat models in the next section.

If one contrasts World War II operations research with today's practice, then it is clear that a major change has occurred in the practice of military OR and the use of models in defense planning. OR has ceased to be a purely scientific discipline, and some, in fact, feel that it has become a purely speculative activity (see, for example, BONDER [9]). During World War II, operations research was primarily concerned with the engineering (i.e. designing and planning) of on-going operations. Consequently, some combat data could be collected as needed for use in studies. Hypotheses about such military operations might actually be scientifically verified by testing against this data. Thus, World War II OR was many times a truly scientific discipline. Today military operations research is primarily concerned with planning of some type; and, as emphasized by BONDER [9], it has ceased to be a truly scientific discipline[12] because of the absence of combat data (see also HOWLAND [46]).

In this vein, SETH BONDER [10] has emphasized that there are almost no empirically verified models of most combat processes. Besides the inherent problem of operational definition and measurement, the major insuperable difficulty in empirically verifying any combat model is that

the historical data base is too poor: it is not rich enough in detail to permit the classic scientific verification of combat models, since nations fight wars for other reasons than to collect combat data. Unfortunately, in the past military historians have been surprisingly reluctant to provide information on battles such as the number of forces of each kind participating on both sides and the losses. H. K. WEISS [115] feels that "the average military historian is particularly susceptible to the criticism aimed by VAGTS [102] (see also [103]) at the 'average military officer' of avoiding 'bellometrics' 'as someting too materialistic and derogatory to military art.'"

This shortage of historical and other empirical data for combat models and analysis is apparently not as widely acknowledged, articulated, or appreciated by the policy-making community (and even some parts of the analysis community) as it should be (see also STOCKFISCH [90]). Moreover, one cannot expect accurate point estimates of combat effectiveness from these models. Rather, such nonempirically developed models should only be used for analysis purposes to provide defense management with [9]:

(R1) insights into directions and trends thereby increasing understanding of the system dynamics,

(R2) guidelines for the development of data-collection plans - what data is important and how accurate it must be,

(R3) guidelines for the development of technological and modelling research plans.

It is in this spirit of developing insights that simplified LANCHESTER-type models of warfare are considered in this book. In the same vein, KARL von CLAUSEWITZ[13] [20, p. 191] stated many years ago in his classic work On War that if theory caused a more critical study of war, then it had achieved its purpose.

Underlying the engineering (i.e. designing and planning) of military operations, evaluation of military systems, and other problems of defense planning, however, should be the scientific study of conflict (in particular warfare). Just as most branches of engineering (for example, mechanical engineering) are based on NEWTONIAN physics, so should military operations research be based on the scientific study of warfare. Unfortunately, appallingly little basic research on conflict and warfare has apparently been conducted.[14] No science of "bellometrics" [102; 115] has as yet emerged. Later in this book we will briefly discuss what has been done with respect to the scientific verification of LANCHESTER-type models of warfare. As mentioned above, the quality and extent of the historical data base have been severely limiting factors for such important investigations.

1.3. Different Types of Combat Models.

As we have discussed in Section 1.1.2. above, models are representations of reality, and we have seen that different types of such representations are possible. With respect to combat operations, Figure 1.1 shows the variety of forms that combat models may take. One can associate trends in model characteristics such as degree of operational realism, abstraction, and convenience and accessibility with this spectrum of combat models. As Figure 1.1 shows us, operational realism and degree of abstraction are conflicting qualities.

For present purposes, let us focus on the three right-most types of combat models depicted in Figure 1.1. Following BONDER [10], we will limit our discussion of combat models to the following three general types:

(T1) war games,

(T2) simulations,

(T3) analytical models.

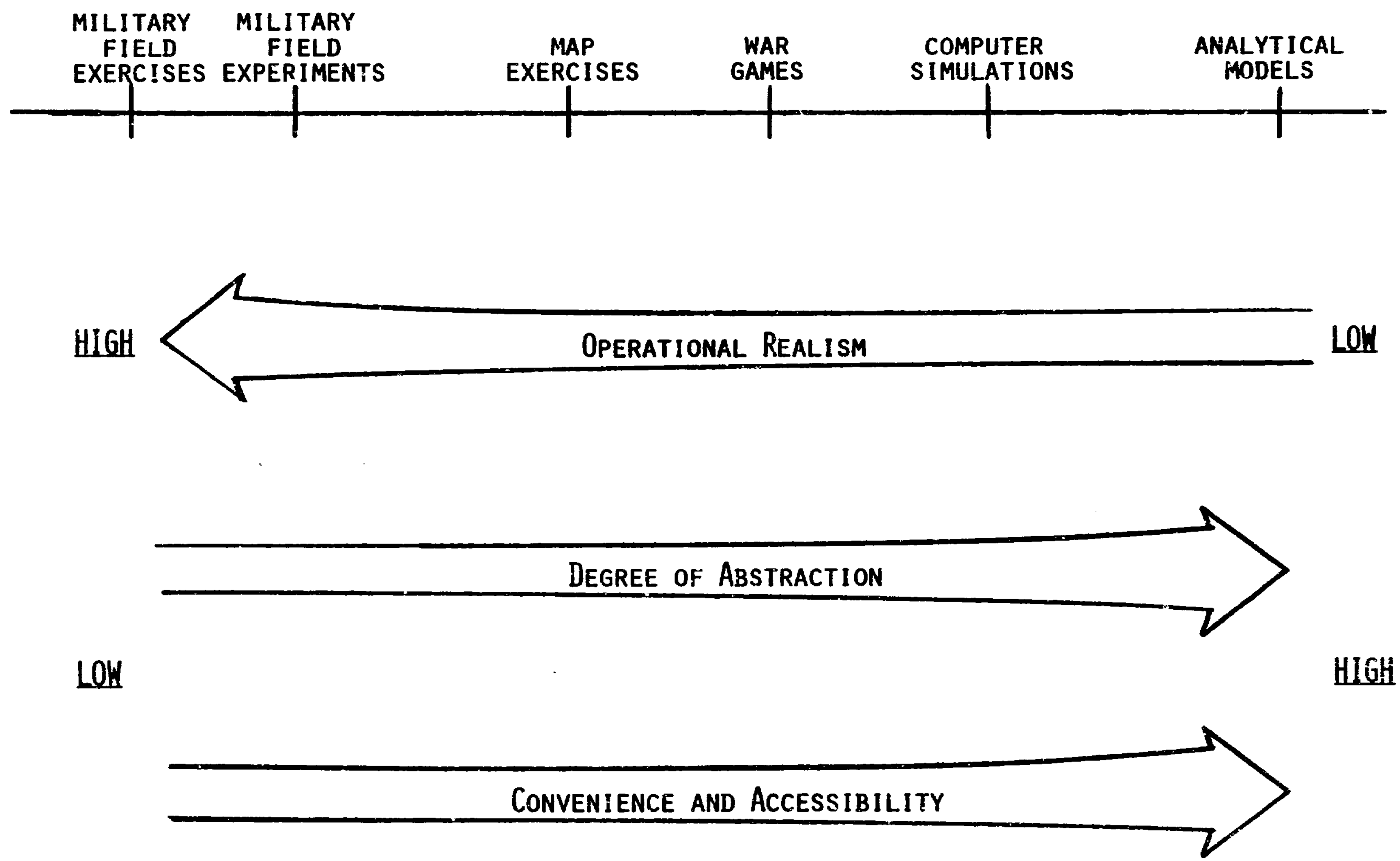


Figure 1.1. The spectrum of types of combat models.

Additionally, in the ensuing discussion we will generally emphasize ground combat models (i.e. models of warfare between ground combat units). Although other classifications are certainly possible, the above is adequate for now.

According to PAXSON [70], "a war game is a model of military reality set up by a judicious process of selection and aggregation, yielding the results of the interactions of opponents with conflicting objectives as these results are developed under more or less definite rules enforced by a control or umpire group." The distinguishing feature of war games in relation to simulations and analytical models, however, is that actual human beings are used to simulate decision processes by having people play the roles of decision makers and use their own judgments in making decisions (see also [42]). This distinction is graphically depicted in Figure 1.2.

War games may be classified as being either "rigid" or "free", depending on whether or not the assessment rules are rigidly prescribed and completely cover all possibilities. These two types of war games (i.e. the rigid and free war games) correspond to the opposing demands of realistic games and playable games. The rigid war games are somewhat similar to simulations in their assessment of combat outcomes in that combat interactions are considered in detail. Before the age of large-scale computers, the sheer immensity of the volume of the details for such rigid assessments was overwhelming: it was not uncommon for many volumes (i.e. books) of rules and combat-results tables to be required for the running of a rigid war game. As a reaction and revulsion to such overwhelming detail, "free" war games were developed, with the assessment of combat outcomes being judgmentally determined by umpires. It is interesting to note that modelling issues such as degree of resolution,

	SIMULATED (Human Element Present)	RIGID DETERMINATION (No Human Element Present)
TACTICAL DECISIONS (Behavioral and Decision Aspects)	WAR GAME	MONTE CARLO SIMULATION ANALYTICAL MODEL

Figure 1.2. Distinction between different types of combat models *according to how* decision making is represented.

appropriate technique of aggregation, amount of detail, etc. were all considered in the past by war gamers of the 19th and 20th centuries.

Today many computer-assisted war games exist, with the computer doing the bookkeeping and assessing combat outcomes. To a certain extent, the modern large-scale digital computer has neutralized some of the shortcomings of rigid war games. Teams of players typically represent the commanding officers and their staffs. However, this type of model, i.e. the rigid (computer-assisted) war game, is very expensive in terms of time and money to develop, maintain, and use. BONDER [10] points out that it typically may take something like four to eight years to develop such a rigid war game. He also notes [10, p. 73] that as recently as 1971 it took six months to obtain one realization of ten hours of battle with a particular war game. War games may be an excellent vehicle for developing general insights and identifying critical elements for further more detailed analysis, but many feel that this type of model is not a feasible vehicle for systematically analyzing a wide variety of system alternatives in a responsive manner [10].

To simulate means to act like. Simulations are models in which processes and activities are "acted out." Systems are microscopically analyzed and modelled by analogue duplication. Because of the large amount of bookkeeping involved in such minute duplication, a large-scale digital computer is a necessity. In fact, the development of the modern digital computer has led to the widespread use of simulation as an analysis technique. Such simulation of combat operations is the modern-day automated version of the classic sand table for military analysis. In essence, such a combat simulation is an analogue model, which recreates the sand table with the help of the digital computer, and battles are acted out on this automated sand table.

Simulation may or may not involve actual human beings playing some

of the decision-making roles in the system modelled. For the purposes of our present discussion, we will limit ourselves to so-called machine simulation that runs on a computer entirely without human participation.[15] Moreover, for convenience we will henceforth refer to machine simulation simply as simulation.

Simulation is probably the most widely used technique for military systems analysis. To develop a simulation of combat operations, the military system and associated activities are microscopically studied and decomposed into a set of basic events, which in turn are ordered in sequence of occurrence (much like a network). When such a model is run to predict combat outcomes such as numbers of casualties of various types, territory lost, resources expended, etc; the battle is essentially "acted out on the computer," with the sequence and flow of events and combat activities followed in the same microscopic sequencing as determined by previous analysis. Human decision making in the combat is simulated with predetermined decision tables or rules.

Moreover, there are some problem areas that are more or less unique to the simulation of combat operations. A major problem area is the representation of terrain, especially the modelling of the line-of-sight process. A high-resolution simulation such as DYNTACS [7; 19] may spend as much as 60 percent of its running time in checking for intervisibility (i.e. the existence of line-of-sight) between weapon systems, and usually at least about 20 percent of its running time is so spent [69]. Thus, an inordinately large amount of time is usually spent in simulating the line-of-sight process in combat simulations. Terrain modelling sometimes receives attention in books on simulation (<u>see</u> EVANS, WALLANCE, and SUTHERLAND [26]), but usually it does not (<u>see</u>, for example, FISHMAN [29]). Other

problem areas (not only for simulation but for combat modelling in general) are the modelling of battlefield intelligence, route selection, and tactical decision processes (especially those relating to the management of large-scale warfare [10]).

Most combat simulations used in defense planning are so-called Monte Carlo simulations because statistical sampling techniques (involving the generation of pseudorandom numbers [29]) are used to determine the outcomes of random events, such as the outcome of firing at a target. Because of the tremendous quantity of computations and other information processing requirements in such a simulation, the use of a modern high-speed digital computer is essential. Probability distributions for all the random elements (i.e. random variables) in the simulation are required as inputs, and consequently a high-resolution Monte Carlo simulation such as DYNTACS requires a rather extensive data base for its running.[16] The difficulties and costs of data base preparation are considerable and are frequently underestimated. The simulation then empirically generates the probability distribution for the set of possible combat outcomes. Each run of the simulation for a given set of input data is essentially a sample from the distribution of outcomes, and the simulation must be run repeatedly to obtain accurate statistical information about this distribution of combat outcomes.

The strong point of Monte Carlo combat simulation is that such a simulation may contain a lot of detail and therefore may be more credible than a more abstract model to many people. Examples[17] of such Monte Carlo simulations are ASARS II, CARMONETTE, DYNTACS, and SIAF. Some people (see SHUBIK and BREWER [86], for example) feel, however, that such simulations make a "fetish of realism." The large amount of detail, moreover, causes a significant amount of computer time to be required for a single run of such a simulation, and this characteristic is essentially their undoing as far as being a viable analysis technique for exploring the limits

of system capability.

There are a number of serious shortcomings to the use of Monte Carlo simulation for defense analysis.[18] First, such simulations are quite costly to build. It is not unreasonable to expect to spend 5 to 10 man-years of effort to develop a detailed simulation of tactical combat.[19] Second, they are costly to run, with typically 10-20 minutes of computer time (IBM 360/67) required per replication of about the same length of battle time, and one needs 10-60 replications for statistical stability in the results (see, for example, ZIMMERMAN [120, p. 741]). Additionally, because of the amount of detail involved, the data-base requirements are quite demanding. For example, it is not unheard of to have several analysts spend about three months preparing a new set of input data and the corresponding data deck for DYNTACS. Not only is a so-called high-resolution combat simulation costly to build and run, but it is also costly to maintain: a staff of fairly highly trained personnel must be maintained to insure that the computer program stays running and debugged as changes are continually implemented. For several reasons (e.g. size of the computer program, complexity of the model, etc.), changes may be quite difficult to implement in such a combat simulation. The tremendous amount of detail (i.e. the large number of variables and other parameters) present in a simulation essentially precludes the running of parametric studies to examine the sensitivity of the model to changes in simulation assumptions and input data. Because of this lack of capability to run parametric studies, it is essentially impossible to use simulation by itself as a vehicle for determining those system capabilities, tactics, and environmental characteristics that significantly influence the system's effectiveness. As S. BONDER points out [11, Chapter 1], simulation is essentially too detailed to be by itself a useful tool for analysis. These disadvantages

of Monte Carlo simulation are summarized in Table 1.I.

Analytical models (like machine simulation) do not involve human participation during running. They may, of course, be either deterministic or stochastic in nature. Their distinguishing characteristic is their degree of abstraction: as Figure 1 shows, analytical models are more abstract than simulations. In fact, a good analytical model is usually quite abstract, poor in the number of variables explicitly considered, but rich in ease of manipulation and clarity of insight [86]. Before the advent of high-speed digital computers, an analytical model consisted of at most a few equations (see LANCHESTER's [51] classic models discussed in Chapter 2). Today large-scale processes and systems can be modelled by many equations with the help of a digital computer. The process under study is analyzed and abstracted (i.e. decomposed into basic events and activities). Then mathematical submodels of events and activities are developed and integrated into an overall structure.

Analytical models of any degree of complexity usually do not yield convenient analytical solutions but require numerical approximation methods and a digital computer for the generation of numerical results. However, in those cases in which an explicit analytical solution can be obtained, one has obviously simplified the process of understanding the model. Insights into the dynamics of combat may be obtained by, for example, examining explicit relations between the independent variables, the model's parameters, and the dependent variables (which are usually related to the MOEs). Such insights are much more difficult to acquire when the solution is not simply expressible in terms of elementary functions and, for example, finite-difference methods must be used to generate numerical (approximate) results, although the model's basic structure is explicitly contained in equations that are readily examined. Thus, although more abstract than simulations,

TABLE 1.I. Disadvantages of Monte Carlo Simulation of Combat

(D1) Costly to build

(D2) Costly to run

(D3) Costly to maintain

(D4) Lack of flexibility for change

(D5) Essentially impossible to perform sensitivity and other parametric studies

analytical models are characterized by their transparency (i.e. ease of revealing their basic structure and assumptions). We will focus on such models in this book.

Analytical models, particularly simple ones, help clarify the relationship between theoretical models, empiricism, and data gathering. An analytical model is usually too simple and restricted to directly solve an actual operational problem. But because of its transparency, the analytical model can warn about potential problem areas, indicate where additional meansuresments are most needed, and identify and order important omissions from the model (see SHUBIK and BREWER [86] for a further discussion).

There is one further general type of combat model that merits our attention, a mixture of two of the above types called the hybrid analytical-simulation model [10]. It has been developed in response to the needs for parametric analysis coupled with the long preparation and run times for Monte Carlo simulations. It combines the strengths of these two modelling approaches by representing some processes in one way and others in the other. Again, the modern high-speed digital computer makes possible the integration of these model types. For example, in battalion-level combat models such as BONDER/IUA (see [92]; also [11; 12]) (and its various derivatives such as BLDM, FAST [13], AMSWAG [36], IHA [104]) and COMAN [18], attrition and target acquisition (and sometimes allocation) processes are modelled analytically, while simulation is used to model battlefield movement processes [10]. The same general approach has been applied to large-scale combat (i.e. combat between division-size and large units) with models such as DIVOPS [106] and VECTOR-2 [107] in which the attrition, maneuver-unit-element and fire-support-sensor acquisition, and terrain-line-of-sight processes are modelled analytically [10]. Such hybrid models use LANCHESTER-type equations (i.e. deterministic differential equations) to represent the

combat attrition process.

A related (but yet distinct) classification of combat models would be according to how they assess the outcomes of tactical engagements (irrespective of how tactical decision making is modelled). Three current approaches for predicting the effectiveness of combat units in such engagements are as follows (see BONDER and FARRELL [11] for further details):

(A1) firepower scores (see STOCKFISCH [90, pp. 6-27]),

(A2) Monte Carlo simulation [33; 120],

(A3) analytical models (e.g. differential equations) [11].

All three approaches have been used to assess the outcomes of combat engagements in war games. We have already discussed Monte Carlo simulation and analytical models above so it remains to discuss the other combat-assessment approach, firepower scores. We will also say some additional words about analytical models in the context of assessing the outcomes of tactical engagements. Finally, we will briefly discuss the relation between the scale of combat operations and these modelling approaches.

The firepower-score[20] approach is basically a technique for aggregating heterogeneous forces (i.e. tanks, artillery, infantry, etc.) into a single homogeneous force on each side. It is an index-number approach, which develops one number (referred to as the firepower index) to represent the "combat potential" of a unit. A linear model is used to develop this index number, i.e. the firepower index, from the scores of individual weapon systems as Table 1.II shows. Moreover, as emphasized by STOCKFISCH [90, p. 7], the words score and index should not be regarded as being synonymous. It is more precise, therefore, to use the term firepower score to refer to the military capability or value of a specific weapon system and to use the term firepower index -- which is obtained by summing scores --

TABLE 1.II. Hypothetical Example of Determination of Firepower Index for a Combat Unit

Weapon Type	Number	Firepower Score	Total Contribution to Firepower Index
Rifle, M-16, 5.56mm	6,000	1	6,000
MG, M-60, .30 cal	150	6	900
MG, M-2, .50 cal	250	10	2,500
Mortar, M-125, 81mm	50	20	1,000
Howitzer, M-109(SP), 155mm	50	40	2,000
Howitzer, 8"	8	30	240
Tank, M60A2	200	100	20,000
TOTAL FIREPOWER INDEX 32,640			

Firepower Index for U.S. Army's 7th Infantry Division

to refer to the military capability or value of some aggregation of diverse weapons. In other words, the firepower-score approach provides a common denominator for aggregating the many different types of weapons on a battlefield, and military combat is characterized by such "combined-arms" operations consisting of many different weapon systems.

How is the basic firepower score for a weapon system determined? There are apparently almost as many different answers to this crucial question as there are different firepower-score methods.[21] Many methods state that the firepower score of a weapon system is essentially the product of a measure of single-round lethality multiplied by the expected expenditure of ammunition during a fixed period of time. Although this procedure appears to yield an objective measure of weapon-system capability, STOCKFISCH [90, pp. 23-78, especially pp. 23-27 and 76-78] points out that actually varying amounts of subjectivity are cranked into various such firepower scores. Moreover, the firepower-score approach probably dates back to World War II, although documentation about it is generally somewhat difficult to come by (see STOCKFISCH [9] for introduction to the scanty firepower-score literature).

In large-scale (i.e. division-level and above) ground-combat models, firepower indices are used as a surrogate for unit strength. They are then in general used to:[22]

(U1) determine engagement outcomes,

(U2) assess casualties,

(U3) determine FEBA movement.

[FEBA stands for Forward Edge of the Battle Area. It is the contact zone between two opposing forces.] The force ratio is the significant factor in such determinations. Here the term force ratio means the ratio of the firepower index (i.e. the aggregation of all the firepower scores in the unit)

of the attacker to that of the defender. Let us consider a hypothetical example to illustrate this point. Consider, for example, the 7th Division of the U.S. Army and assume that the firepower scores shown in Table 1.II apply. Then the 7th Division has a firepower index of 32,640. If an attacking enemy Army Group were to have a firepower index of 146,880, then we would have a force ratio of 4.50 (A/D), where A refers to the attacker and D to the defender.

Although the firepower-score approach has been widely used for top-level planning, it has received increasing criticism in recent years (see, for example, STOCKFISCH [90] or [11]). Significant deficiencies of the index-number approach are the following (from [11]):

(D1) it does not measure the accomplishment of unit missions,

(D2) it ignores most of the significant factors that affect mission accomplishment (i.e. weapon system characteristics, threat variables, organizational structures, tactics employed, environmental conditions, etc.),

(D3) it oftentimes bears little relation to the physical combat or other processes under study.

STOCKFISCH [90, p. 128] claims that no satisfactory simple technique for aggregating modern conventional forces currently exists. Although the firepower-score approach has been thus far much criticized, conventional forces must be aggregated in many analyses, and until a better alternative is developed, firepower scores will continue to be used.

Analytical models have been discussed in general terms above. We will now discuss their use specifically for assessing the outcomes of combat engagements. In particular, differential-equation models have been fairly widely used for the assessment of combat outcomes. Such models are

used to represent the decay in numbers of weapon systems (i.e. the attrition process) and require submodels (again usually analytical ones) for various subordinate processes such as target detection, target location, fire allocation, etc. The modern large-scale digital computer has made possible the development of large-scale hierarchical system models, with submodels feeding information into a master coordinating model. In the field of combat modelling, the basic calculation is one of force attrition, and consequently is usually done with the aid of some type of differential-equation model. The use of such models as practical analysis tools is primarily due to the efforts of S. BONDER and his colleagues formerly at the University of Michigan and now at Vector Research, Inc. Their main contribution has been the development of fairly detailed submodels for the prediction of loss rates from engineering and operational data for such differential-equation models. We will refer to such a differential-equation model that represents attrition from enemy action through a system of differential equations for the force levels as a LANCHESTER-type model of warfare (also commonly called a differential combat model [16]). The rest of this book concerns such models.

Each of the above combat-assessment approaches (i.e. firepower scores, Monte Carlo simulation, and analytical models) may be thought of as corresponding to a different scale of combat operations, with the firepower-score approach and Monte Carlo simulation being at opposite ends of the spectrum of the scale of combat operations (i.e. the size of the units involved). This correspondence is shown in Table 1.III. The contents[23] of Table 1.III are only generally true, with exceptions certainly existing. As we see from this table, the firepower-score approach has been primarily used for engagement assessments in large-scale (i.e. theater-level) combat

TABLE 1.III. Combat-Assessment Approach Related to Scale of Combat Operations

Modelling Approach	Scale of Combat - Example[23] of Model
firepower score	theater - ATLAS, CEM
Monte Carlo simulation	infantry: platoon - ASARS II armor: company/battalion - DYNTACS, CARMONETTE
LANCHESTER-type model	battalion - BONDER/IUA division - DIVOPS theater - VECTOR-2, TWSP, BALFRAM, DMEW

models. Although there are exceptions, high-resolution Monte-Carlo simulation has been a feasible assessment approach only when there have been no more than about 100 elements (e.g. individual tanks, crew-served weapons, etc.) on each side. On the other hand, LANCHESTER-type models have been developed for the full spectrum of combat operations, from combat between company/battalion-sized units to theater-level combat operations.

1.4. The Influence of Modern-Digital Computer Technology.[24]

Without the modern high-speed digital computer both high-resolution Monte Carlo simulations such as DYNTACS and CARMONETTE and also differential combat models such as BONDER/IUA and its many derivatives would be impossible. The modern computer provides not only large-scale memory capacity but also the ability to perform millions of arithmetic operations per second. In such a computational environment, the numerical integration of a system of hundreds of ordinary differential equations becomes possible. Today LANCHESTER-type complex system models, which rely on modern digital computer technology for their implementation (see, for example, BONDER and HONIG [12]), have been developed for various levels of combat, from combat between battalion-sized units (see BOSTWICK et al. [13] or HAWKINS [36]) to theater-level operations (see CORDESMAN [21], FARRELL [28], or [105; 107]).

1.5. The Purpose of This Book.

As indicated above, there currently appears to be a trend toward increasing interest in LANCHESTER-type models of warfare. However, information about the nature of such models, their strengths and weaknesses, etc., unfortunately does not appear to be widely disseminated beyond a relatively small group of research workers. Moreover, there have been essentially no readily

accessible sources of general information about LANCHESTER-type models: there has been no book, textbook, or monograph on LANCHESTER-type models of warfare, and the one and only survey article by DOLANSKY [23] appeared in 1964. Considering contemporary developments, DOLANSKY's article is quite out of date today. Furthermore, results and developments have been widely scattered in the literature, and it has been difficult (if not impossible) for an analyst to obtain general information and an overview of LANCHESTER-type models of warfare.

The purpose of this book is to provide a comprehensive survey of LANCHESTER-type models of warfare. By LANCHESTER-type models of warfare we mean differential-equation models that describe changes over time in the force levels of the combatants and other significant variables that describe the combat process. Our objective is to present a unified treatment of such models and of their behavior, with emphasis on the insights that may be consequently obtained into the dynamics of combat. We hope to tie together much of the knowledge about LANCHESTER-type models that has been heretofore widely scattered in the literature.

In the past (say up until about 1970), LANCHESTER-type models of warfare were only used by a small group of the leading analysts: as a consequence of pioneering work by F. W. LANCHESTER[25] [51] done about the time of World War I, a few military operations analysts have used simplified deterministic[26] differential-equation models to develop insights into the dynamics of combat from about the end of World War II (see, for example, [8; 11; 12; 23; 94; 110-112]). The advent of the modern high-speed digital computer has made feasible the development and use of quite complicated versions of such LANCHESTER-type (also frequently called differential) models as practical defense planning tools [10]. Thus, today militarily

realistic computer-based LANCHESTER-type models of quite complex combat systems have been developed and are fairly widely used by a much larger number of analysts than ever used the simple differential-equation models. Thus, the modern digital computer has made much more extensive use of these models possible. Such models currently exist for almost the entire spectrum of combat operations, from combat between battalion-sized [13] and division-sized [16] units to theater-level operations [21; 28]. The study of the basic nature and behavior of such differential combat models is the subject of this book. Our goal is to promulgate a better understanding of such models.

Two divergent aspects of LANCHESTER-type combat models are the following:

(A1) insights that they provide into the dynamics of combat,

(A2) their enrichment in order to better model real-world combat activities.

As is always the case, a book reflects the tastes and interest of its author. Inspired by the works of F. W. LANCHESTER and H. K. WEISS, I have been more interested in obtaining insights into the dynamics of combat from relatively simple models than enriching such models in details (see W. T. MORRIS [63] for a discussion of the processes of such enrichment). Hence, this book emphasizes studying relatively simple combat models in order to learn their basic nature and to, hopefully, perceive significant interrelationships that are difficult to discern in more complex models. Such insights can provide valuable guidance for more detailed computerized investigations (see WEISS [112]). We will also consider the use of LANCHESTER-type models of warfare for developing quantitative insights into optimal time-sequential combat strategies (see Chapter 8).

1.6. Dynamic Systems and State Variables.

The LANCHESTER-type combat models considered in this book may be viewed from the vantage point of system theory (see PADULO and ARBIB [68]). We will find it convenient to do so in order to better understand the philosophical underpinnings of such models. Let us therefore introduce the reader to some intuitive notions and ideas related to systems. We will not attempt to give explicit and precise definitions. For our purposes intuitive and rather vague terminology will suffice.[27]

A physical system is defined as an interconnection of physical elements, or objects. The notion of a system is rather broad: it applies not only to simple mechanical and electrical devices but also to more esoteric and complex systems such as automobiles and (especially) weapons systems. In particular, one can view military units such as companies and battalions as systems.

Systems may be either static or dynamic. This book concerns dynamic combat systems. For our purposes, a dynamic system is one whose inputs and outputs are related by a set of differential (or difference) equations. The system evolves dynamically over time. The set of differential equations provides a model for the system's evolution. We require that such a model be valid in the sense that the present predicts the future. Let us informally, therefore, introduce the notion of cause and effect or, more formally, the principle of casualty. Consider the following example: in NEWTONIAN mechanics, the future motion of a system of particles is completely determined if the present positions and moments are known, along with the present and future forces. Future forces have no affect on the present (nonanticipatory system), and how the system reached its present state is not important.

Knowledge of the present allows us to predict the future. What we must know about the present (besides the equations that describe the evolution of such quantitites) is called tha state of the system. Intuitively, the state of a system is the minimum amount of present information about the history of the system that allows one to predict the effect of the past upon the future. The variables that are used to describe the state of a system are called the state variables.

The above terminology is convenient for communication about LANCHESTER-type models of warfare. Later when we consider time-sequential combat strategies, it will be convenient to introduce the system-theory notions of closed-loop and open-loop controls. As we will see in the next chapter, one may view LANCHESTER's classic combat theory as saying that force levels are the state variables for combat between two military systems. We return to this theme later.

1.7. Final Remarks.

Thus, we see that we may say that LANCHESTER-type models of warfare represent dynamic combat systems whose state variables are typically force levels. In this introductory chapter we have established a framework for studying such differential-equation models of combat: we have examined the general nature of models, the use of combat models in defense planning in the United States, and the various types of combat models that are in current use. Based on our examination of the scientific study of conflict and warfare, we feel that most of the shortcomings usually attributed to LANCHESTER-type models[28] are also the shortcomings of any combat model.

Moreover, we feel that LANCHESTER-type models are an ideal vehicle for studying combat dynamics because of the relative ease of extracting information from them and the fact that usually no other type of model is better justified.

Our conclusion is based on a careful examination of the state-of-the-art of conflict and combat modelling. In the next chapter we will see how LANCHESTER-type models readily provide many important insights into the dynamics of combat.

FOOTNOTES for Chapter 1

1. Unfortunately, little of this debate has reached the open literature. See, however, the excellent report by the U.S. Army Models Review Committee [42], BONDER and FARRELL [11, Chapter 1], and BONDER [10].

2. For some differing views on the nature of operations research, see BARISH [4], BONDER [9], CHURCHMAN, ACKOFF, and ARNOFF [17], GOODEVE [34], KLEIN and BUTKOVICH [50], MISER [59; 60], and references contained therein.

3. Although this definition opens the classic book by MORSE and KIMBALL [64], the definition apparently goes back to KITTEL [49] (as reported by GOODEVE [35]).

4. See, for example, MORSE and KIMBALL [64], CHURCHMAN, ACKOFF, and ARNOFF [17], HILLIER and LIEBERMAN [40], or WEISS [113]. See also the references cited in Footnote 2.

5. Here the letter C is used phonetically to denote that we are enumerating concepts in this list. For the next such enumeration in this book, the letter T is used to denote that we are listing types (of models).

6. The effectiveness of any military system may be defined as the extent to which the system may be expected to achieve a set of objectives [109], and the quantitative expression of the extent to which specific mission requirements are attained by the system is referred to as a measure of effectiveness (MOE). In OR work, it is important to distinguish between the performance (e.g. rounds fired per minute, single shot kill probability,

etc.) of a weapon system and its effectiveness (e.g. decisively winning a fire fight), or military worth. Failure to choose appropriate measures of effectiveness can lead to completely wrong conclusions as to preferred alternatives (see MORSE and KIMBALL [64]). As stated in the main text, although performance data for a weapon system may be collected in "operational" tests, a combat model is usually required (for example, due to safety considerations) to "put it all together" against an enemy threat in an operating environment to estimate system effectiveness (see, for example, RUDWICK [80, p. 57]). In other words, the combat model transforms performance measures (e.g. target acquisition capability, rate of fire, etc.) into effectiveness measures (e.g. battle outcome, FEBA movement).

7. About $30 to $40 million is apparently spent each year for just the construction of such models. Unfortunately, it is very difficult to estimate how much money is actually being spent annually on combat modelling activities because of the nonexistence of cost-accounting definitions and procedures [86].

8. See, for example, QUADE and BOUCHER [74, pp. 221-225].

9. In the decision sciences, the word "uncertainty" has a special technical meaning (see, for example, LUCE and RAIFFA [54]). However, we are using this word as being synonymous with "having an element of chance involved."

10. See, for example, HITCH and McKEAN [41], QUADE [73], ENKE [25], QUADE and BOUCHER [74], or BONDER [9].

11. Here we are brought face to face with the disagreeable paradox pointed out by M. HOWARD [45, p. 10] that "war might be necessary as an instrument of policy to insure the survival of a society in which it was possible to renounce war as an instrument of policy." Speaking about World War II, he went on to say [45, pp. 10-11], "Good will and international organizations were apparently not enough in themselves to eliminate violence as an element in international affairs." In the mid-1960's and early 1970's a wave of sentiment (remarkably similar to that reported by HOWARD [45, p. 10] for post-World-War-I England) arose within American academe (and especially within the OR community) that war was not a problem to be examined but an evil to be shunned. The parallel with the intellectual climate of the 1920's and 1930's (as reported by HOWARD) is uncanny.

12. There is a special problem which has gone largely unnoticed, for those who wish to test the validity of models of defense/military systems and/or operations: the data base for the testing of such a model is from the real world (past and present), whereas the prediction from the model is for the real world (future). The physical sciences are based on the principle of <u>uniformitarianism</u>, which holds that physical and biological processes, conditions, and operations do not change over time (i.e. uniformity over time). For example, in geology the doctrine of uniformitarianism holds that the present is the key to the past [61]. This principle, of course, does not hold for planning models of new future environments (<u>see</u>, for example, HOWLAND [46]). What is meant by the validity of such a planning model is in need of critical examination.

13. For a discussion of von CLAUSEWITZ and the other major writer of the NAPOLEONIC age on the art of war (namely, General Baron de JOMINI), <u>see</u> EDMONDS [24].

14. Concerning the scientific study of warfare, let us note some of the work that has been done in the fields of arms races and warfare in general. LEWIS FRY RICHARDSON did pioneering work in both fields [78; 79]. For a lucid and authoritative discussion of RICHARDSON's mathematical theory of war (including arms races), see RAPOPORT [75]. For an introduction to the scientific study of arms races, see INTRILIGATOR and BRITO [47], RATTINGER [76], SAATY [81] and WEISS [113]. H. K. WEISS [114] has pointed out that although more books have been written about war than about almost any other human experience, the number of quantitative analyses is extremely small. The most notable of these are the pioneering studies by QUINCY WRIGHT [117] and L. F. RICHARDSON [79].

SAATY [81] points out that in 1965 a Norwegian statistician used a computer to organize a data base for 14,531 wars in 5,560 years of recorded history. This data suggests that RICHARDSON's [79] pioneering quantitative study of 315 wars that ended between 1800 and 1952 may well be representative of the entire recorded history of man on earth. H. K. WEISS [114] has taken RICHARDSON's data as a point of departure for developing several stochastic models for the duration and magnitude of wars. HORVATH [44], however, has criticized this work and suggested an alternative model based on the theory of extreme values. All this data suggests that unfortunately, war has been quite an established human institution. Moreover, the author feels that one should view the scientific study of war (including Lanchester-type and other combat models) much as one views the study of, for example, a disease like cancer: the subject area may be unpleasant but somebody must understand the phenomenon to be able to realistically suggest what to do about it.

15. One, for example, develops simple decision tables or rules to model the

complex human decision-making process.

16. However, Monte Carlo combat simulations are not appreciably more demanding in their input requirements than detailed hybrid analytical-simulation combat models such as BONDER/IUA and its various derivatives discussed below.

17. Even when it exists, documentation of a combat model may be poor [86]. However, the following documentation and information is exceptionally good for this field. Further information about CARMONETTE may be found in ZIMMERMAN [120] or ADAMS, FORRESTER, KRAFT, and OOSTERHOUT [3]. CARMONETTE was an early effort in ground combat simulation and won the Lanchester Prize (see Footnote 24) for RICHARD E. ZIMMERMAN [119] in 1956. Further information about DYNTACS is to be found in [7; 19], while that about SIAF is in [99]. General information about current combat models (mainly Monte Carlo simulations) is available in [92; 101].

18. Our discussion here follows BONDER [10].

19. CARMONETTE, a pioneering combat simulation, took about 20 man-years of effort to develop [3, p. 6]. For more recent data on the cost of simulation development, see SHUBIK and BREWER [86].

20. Indices of the relative combat capabilities of military units (based on a "scoring system" for the weapons employed in the units) have been used by military gamers and force planners for years. We are here generically referring to such indices as firepower scores, i.e. using the term firepower scores to refer to any one of a large family of such indices. Other

members of this family of indices and related terms are firepower potential (FP), firepower potential score (FPS), unit firepower potential (UFP), index of firepower potential (IFP), index of combat effectiveness (ICE), weapon effectiveness index/weighted unit value (WEI/WUV), weapon effectiveness value (WEV), etc. (see STOCKFISCH [90, pp. 6-27] for further references and a guide to the literature about firepower scores).

21. Names of various firepower-score methods are given in Footnote 20. See STOCKFISCH [90] for further information.

22. The exact details vary from model to model. Sometimes (U1) and (U2) are combined.

23. As pointed out in Footnote 17, documentation of combat models is generally poor. The following documentation and information is, however, exceptionally good for this field. General information about contemporary combat models (mainly Monte Carlo simulations) is available in [92; 101]. Further information about ATLAS may be found in KERLIN and COLE [48] or [33], while that about CEM may be found in [15] or [53]. Documentation of both DYNTACS and CARMONETTE has been discussed above in Footnote 17. Information about BONDER/IUA and its various derivative models may be found in [11; 12; 36; 92; 104], while that about DIVOPS may be found in [106]. The theater-level combat model named VECTOR is documented in [21; 105; 107]. DMEW (see [100]) is also a theater-level model, as is TWSP (see [21] or [27]).

24. For an excellent general discussion of computers and national security, see PAXSON [71].

25. FREDERICK W. LANCHESTER (1868-1946) was an eminent English automotive and aeronautical engineer. For a brief sketch of his many scientific and engineering contributions, *see* McCLOSKEY [55]. The Lanchester Prize is named after him and is awarded annually by the Operations Research Society of America "for the paper on operations research judged to be the best of the calendar year."

26. Corresponding stochastic formulations (i.e. Markov-chain analogues) are for all practical purposes analytically intractable (*see* Note 1 of TAYLOR and BROWN [93, p. 65]).

27. *See* PADULO and ARBIB [68] or TIMOTHY and BONA [98] for more precise and extensive discussions.

28. *See*, for example, the shortcomings given in Section 2.6 for LANCHESTER's classic (constant-coefficient) combat formulations.

NOTES and REMARKS for Chapter 1

Our discussion of models in Section 1.1.2 is similar to that of ACKOFF [2, Chapter 4]. Further discussion in a similar vein is to be found in CHURCHMAN, ACKOFF, and ARNOFF [17, Part III] Our discussion of the different types of combat models in Section 1.3 owes much to BONDER and FARRELL [11, Chapter 1] and BONDER [9; 10].

World-War-II Operations Research. Further information about World-War-II operations-research activities may be found in McCLOSKEY and TREFETHEN [57] and McCLOSKEY and COPPINGER [56]. For some idea about the subsequent development of OR, see (for example) DAVIES, EDDISON, and PAGE [22], ACKOFF [1], HERTZ and EDDISON [39], and any recent textbook on OR (see, for example, the fairly extensive references given in WAGNER [10]). The book by STOCKFISCH [89] contains not only a very good description of World-War-II OR activities but also an outstanding description and analysis of the subsequent development and use of OR, cost-effectiveness analysis, and their many variants by DoD.

Defense Planning. For discussions (the classic ones) of defense planning, see HITCH and McKEAN [41], ENKE [25], QUADE [73], and QUADE and BOUCHER [74]. For an older account of the weapons-acquisition process, see PECK and SCHERER [72]. Overall discussion of American defense policy is to be found in HEAD and ROKKE [37]. Information about the yearly Planning-Programming-Budgeting-System (PPBS) Cycle and its evolution is to be found in ENKE [25] and NOLAN [67]. STOCKFISCH [89] has given a penetrating analysis of weapon-system development and procurement by DoD. He has postulated flaws that lead to the military bureaucracies operating under "perverse incentives" in the cur-

rent defense system, and he has also made suggestions for improving DoD management (see also STOCKFISCH [90; 91]). For discussions of contemporary defense-policy issues, see various publications of The Brookings Institution (for example, LAWRENCE and RECORD [52], or RECORD [77]). Issues for the fiscal year 1977 are discussed in SCHNEIDER and HOEBER [82].

Systems Analysis. For various views on the nature of systems analysis, its role in defense planning, and its relationship to OR, see QUADE [73], QUADE and BOUCHER [74], RUDWICK [80], and NOLAN [67]. For a critical discussion of systems analysis in nonmilitary contexts, see HOOS [43]. In fact, the study of "systems" has become quite a field of study in its own right (see, for example, von BERTALANFFY [6]). Unlike the variety of systems analysis practiced in the defense community (see the above references [67; 73-74; 80]), the brand of systems theory espoused by von BERTALAFFY and others of this general school of systems science (see, moreover, HOOS [43, pp. 15-41] for a brief and penetrating survey of the diverse meanings of the word "system" as used in many different disciplines) uses differential-equation models as the basic vehicle for studying the dynamical behavior of systems. In this respect, see (for example) the work of FORRESTER [30-32]. Moreover, FORRESTER's work, in contrast to the work at hand, has stressed an "experimental" approach to understanding system behavior through the repeated running of continuous-time simulations (i.e. numerical integration of systems of differential equations, not Monte Carlo simulation). This work has not been without its critics, though (see, for example, SHUBIK [83], BREWER and HALL [14], and BERLINSKI [5]). Moreover, the analogue in the defense community of FORRESTER's work has been that of PAUL CHAIKEN of the Stanford Research Institute (see, for example, [58]).

Simulation and Gaming. For an early general account of simulation, see MORGENTHALER [62]. More recent accounts are contained in, for example, the books by NAYLOR, BALINTFY, BURDICK, and CHU [66], EVANS, WALLACE, and SUTHERLAND [26], and FISHMAN [29]. The latter book [29] (see also NAYLOR [65]) contains fairly extensive references to the simulation literature. Most of this literature, however, is irrelevant to our current examination of combat models and defense planning: a very small portion of the contemporary literature on simulation (one exception being the book by EVANS, WALLACE, and SUTHERLAND [26]) considers the simulation of military combat or other military operations and is therefore relevant to the analysis of defense-planning problems. Along these lines, ZIMMERMAN's 1960 article [119] is probably still the best article available on the simulation of ground combat. Although the list of combat simulations that we have given above (see, for example, Footnote 17) is rather short, it does include most of the principal ones that are being used by DoD today.

We probably have not done justice to the topic of gaming. For recent general discussions of various aspects of gaming, see SHUBIK [84; 85] (see also SHUBIK and BREWER [87] and SHUBIK, BREWER, and SAVAGE [88]). The latter book [85] contains excellent guides to various parts of the gaming literature. For a very readable and informative popular account of war gaming, see WILSON [116]. We agree, moreover, with SHUBIK and BREWER [86, p. 8] that "the amount of publicity given free-form, political-diplomatic-military games has been enormously disproportionate to the financial and intellectual investments in them. Popular accounts aside (such as [116]), research on the intellectual foundations and uses of this type of work has been neglible." The classic work on "traditional" war gaming is by YOUNG [118] and contains a comprehensive history of the development of war gaming. For accounts of operational gaming and its role in military operations research, see THOMAS

and DEEMER [97], THOMAS [95; 96], and PAXSON [70]. Although somewhat dated, the references [95-97] are still an excellent introduction to gaming, probably the best technical one in the military field. A more recent version of this material (but not as deep or comprehensive in the military area) is to be found in the book by SHUBIK [84].

REFERENCES for Chapter 1

1. R. L. Ackoff (Editor), Progress in Operations Research, Volume I, John Wiley, New York, 1961.

2. R. L. Ackoff, Scientific Method: Optimizing Applied Research Decisions, John Wiley, New York, 1962.

3. H. Adams, R. Forrester, J. Kraft, and B. Oosterhout, "CARMONETTE: A Computer-Played Simulation," ORO-T-389, Operations Research Office, The Johns Hopkins University, Chevy Chase, Maryland, February 1961.

4. N. N. Barish, "Operations Research and Industrial Engineering: The Applied Science and Its Engineering," Opns. Res. 11, 387-398 (1963).

5. D. Berlinkski, On Systems Analysis, The M.I.T. Press, Cambridge, 1976.

6. L. von Bertalanffy, General System Theory, George Braziller, New York, 1968.

7. A. Bishop and G. Clark, "The Tank Weapon System-Management Summary," Report No. RF 573 AR 69-4, Systems Research Group, The Ohio State University, Columbus, Ohio, June 1969.

8. S. Bonder, "A Theory for Weapon System Analysis," Proc. U.S. Army Opns. Res. Symp. 4, 111-128 (1965).

9. S. Bonder, "Systems Analysis: A Purely Intellectual Activity," Military Review 51, No. 2, 14-23 (1971).

10. S. Bonder, "An Overview of Land Battle Modelling in the U.S.," Proc. U.S. Army Opns. Res. Symp. 13, 73-88 (1974).

11. S. Bonder and R. Farrell (Editors), "Development of Models for Defense Systems Planning," Report No. SRL 2147 TR 70-2, Systems Research Laboratory, The University of Michigan, Ann Arbor, Michigan, September 1970 (AD 714 677).

12. S. Bonder and J. Honig, "An Analytic Model of Ground Combat: Design and Application," Proc. U.S. Army Opns. Res. Symp. 10, 319-394 (1971).

13. S. Bostwick, F. Brandi, C. Burnham, and J. Hurt, "The Interface Between DYNTACS-X and Bonder-IUA," Proc. U.S. Army Opns. Res. Symp. 13, 494-502 (1974).

14. G. Brewer and O. Hall, "Policy Analysis by Computer Simulation: The Need for Appraisal," P-4893, The RAND Corporation, Santa Monica, California, August 1972.

15. J. Bruner and 11 other authors, "Theater Force Performance in CONAF II, Volume 1 - Methodology, Part 1," OAD-CR-1, General Research Corporation, McLean, Virginia, May 1973.

16. W. P. Cherry, "The Role of Differential Models of Combat in Fire Support Analysis," Appendix 4 in Fire Support Requirements Methodology Study Phase II, Proceedings of the Fire Support Methodology Workshop, R. M. Thackeray (Editor), Ketron, Inc., Arlington, Virginia, December 1975.

17. C. W. Churchman, R. L. Ackoff, and E. L. Arnoff, Introduction to Operations Research, John Wiley, New York, 1957.

18. G. Clark, "The Combat Analysis Model," Ph.D. Thesis, The Ohio State University, Columbus, Ohio, 1969.

19. G. Clark, S. Parry, D. Hutcherson, and J. Rheinfrank, "Small Unit Combat Simulation (DYNTACS X)," Report No. RF 2978-FR 71, Systems Research Group, The Ohio State University, Columbus, Ohio, March 1971.

20. C. von Clausewitz, On War, edited with an introduction by A. Rapoport, Penguin Books, Ltd., Harmondsworth, Middlesex, England, 1968.

21. A. Cordesman (Editor), "Developments in Theater Level War Games," unpublished materials for C-5 Working Group of 35th Military Operations Research Symposium, 1975.

22. M. Davies, R. Eddison, and T. Page (Editors), Proc. First International Conference on Operational Research, Operations Research Society of America, Baltimore, 1957.

23. L. Dolansky, "Present State of the Lanchester Theory of Combat," Opns. Res. 12, 344-358 (1964).

24. J. E. Edmonds, "Jomini and Clausewitz," Army Quarterly 62, 101-106 (1951).

25. S. Enke (Editor), Defense Management, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

26. G. Evans, G. Wallace, and G. Sutherland, Simulation Using Digital Computers, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

27. W. Fain, J. Fain, and H. Karr, "A Tactical Warfare Simulation Program," Naval Res. Log. Quart. 13, 413-436 (1966).

28. R. Farrell, "VECTOR 1 and BATTLE: Two Versions of a High-Resolution Ground and Air Theater Campaign Model," pp. 233-241 in Military Strategy and Tactics, R. Huber, L. Jones, and E. Reine (Editors), Plenum Press, New York, 1975.

29. G. Fishman, Concepts and Methods in Discrete Event Digital Simulation, John Wiley, New York, 1973.

30. J. Forrester, Industrial Dynamics, The M.I.T. Press, Cambridge, Massachusetts, 1961.

31. J. Forrester, Urban Dynamics, The M.I.T. Press, Cambridge, Massachusetts, 1969.

32. J. Forrester, World Dynamics, Wright-Allen Press, Inc., Cambridge, Massachusetts, 1971.

33. General Research Corporation, "A Hierarchy of Combat Analysis Models," McLean, Virginia, January 1973.

34. C. F. Goodeve, "Operational Research," Nature 161, 377-384 (1948).

35. C. F. Goodeve, "Operational Research in the Research Associations," Nature 161, 584-585 (1948).

36. J. Hawkins, "The AMSAA War Game (AMSWAG) Computer Combat Simulation," AMSAA Tech. Report No. 169, U.S. Army Material Systems Analysis Activity, Aberdeen Proving Ground, Maryland, July 1976.

37. R. G. Head and E. J. Rokke, American Defense Policy, Third Edition, Johns Hopkins Press, Baltimore, 1973.

38. C. C. Herrmann and J. F. Magee, "Operations Research for Management," Harvard Business Rev. 31, 100-112 (1953) (reprinted on pp. 1-29 of New Decision-Making Tools for Managers, E. C. Bunsk and J. F. Chapman (Editors), Harvard University Press, Cambridge, Massachusetts, 1963).

39. D. B. Hertz and R. T. Eddison (Editors), Progress in Operations Research, Volume II, John Wiley, New York, 1964.

40. F. Hillier and G. Lieberman, Introduction to Operations Research, Holden-Day, Inc., San Francisco, 1967.

41. C. Hitch and R. McKean, The Economics of Defense in the Nuclear Age, Harvard University Press, Cambridge, 1967.

42. J. Honig, R. Blum, H. Holland, D. Howes, D. Lester, K. Myers, and R. Zimmerman, "Review of Selected Army Models," Assistant Vice Chief of Staff (Army), Washington, D.C., May 1971 (AD 887 175).

43. I. R. Hoos, Systems Analysis in Public Policy, University of California, Berkeley, 1972.

44. W. J. Horvath, "A Statistical Model for the Duration of Wars and Strikes," Behavioral Sci. 13, 18-28 (1968).

45. M. Howard, Studies in War and Peace, The Viking Press, New York, 1970.

46. D. Howland, "The Trade-Off Problem in Weapon System Design," Military Review 43, No. 10, 72-78 (1963).

47. M. Intriligator and D. Brito, "Formal Models of Arms Races," J. Peace Sci. 2, 77-88 (1976).

48. E. Kerlin and R. Cole, "ATLAS: A Tactical, Logistical, and Air Simulation; Documentation and User's Guide," RAC-TP-338, Research Analysis Corporation, McLean, Virginia, April 1969.

49. C. Kittel, "The Nature and Development of Operations Research," Science 105, 150-153 (1947).

50. D. Klein and P. Butkovich, "Can the Professions of Operations Research Management Science Change and Survive," Interfaces 6, 47-51 (1976).

51. F. W. Lanchester, "Aircraft in Warfare: The Dawn of the Fourth Arm - No. V., The Principle of Concentration," Engineering 98, 422-423 (1914) (reprinted on pp. 2138-2148 of The World of Mathematics, Vol. IV, J. Newman (Editor), Simon and Schuster, New York, 1956).

52. D. Lawrence and J. Record, U.S. Force Structure in NATO: An Alternative, The Brookings Institution, Washington, D.C., 1974.

53. P. Louer and 5 other authors, "Conceptual Design for the Army in the Field Alternative Force Evaluation, CONAF Evaluation Model IV, Part I - Model Description," OAD-CR-60, General Research Corporation, McLean, Virginia, September 1974.

54. R. Luce and H. Raiffa, Games and Decisions, John Wiley, New York, 1957.

55. J. McCloskey, "Of Horseless Carriages, Flying Machines, and Operations Research: A Tribute to Frederick William Lanchester (1868-1946)," Opns. Res. 4, 141-147 (1956).

56. J. F. McCloskey and J. M. Coppinger, Operations Research for Management, Volume II, Johns Hopkins Press, Baltimore, 1956.

57. J. F. McCloskey and F. N. Trefethen, Operations Research for Management, Johns Hopkins Press, Baltimore, 1954.

58. E. H. Means et al., "BALFRAM User Manual for the Staff of the Commander in Chief Pacific," NWRC-TN-52, Stanford Research Institute, Menlo Park, California, September 1974.

59. H. J. Miser, "The Environment for Effective Operations and Systems Research," Management Science 10, 1-7 (1963).

60. H. J. Miser, "What is Operations Research," Opns. Res. 22, 903-909 (1974).

61. R. Moore, Introduction to Historical Geology, Second Edition, McGraw-Hill, New York, 1958.

62. G. W. Morgenthaler, "The Theory and Application of Simulation in Operations Research," pp. 363-419 in Progress in Operations Research, Volume I, R. Ackoff (Editor), John Wiley, New York, 1961.

63. W. T. Morris, "On the Art of Modelling," Management Sci. 13, B-707 - B-717 (1967).

64. P. Morse and G. Kimball, Methods of Operations Research, The M.I.T. Press, Cambridge, Massachusetts, 1951.

65. T. Naylor, "Simulation and Gaming, Bibliogr. 19," Computing Reviews 10, 61-69 (1969).

66. T. Naylor, J. Balintfy, D. Burdick, and K. Chu, Computer Simulation Techniques, John Wiley, New York, 1966.

67. R. Nolan, "Systems Analysis and Planning-Programming-Budgeting Systems (PPBS) for Defense Decision Making," Naval Res. Log. Quart. 17, 359-372 (1970).

68. L. Padulo and M. Arbib, System Theory, W. B. Saunders Company, Philadelphia, 1974.

69. S. H. Parry, private communication, January 1977.

70. E. W. Paxson, "War Gaming," RM-3489-PR, The RAND Corporation, Santa Monica, California, February 1963.

71. E. W. Paxson, "Computers and National Security," pp. 65-92 in Computers and the Problems of Society, H. Sackman and H. Borko (Editors), AFIPS Press, Montvale, New Jersey, 1972.

72. M. Peck and F. Scherer, The Weapons Acquisition Process: An Economic Analysis, Graduate School of Business Administration, Harvard University, Boston, 1962.

73. E. S. Quade (Editor), "Analysis for Military Decision," R-387-PR, The RAND Corporation, Santa Monica, California, November 1964.

74. E. S. Quade and W. Boucher (Editors), Systems Analysis and Policy Planning, Elsevier, New York, 1968.

75. A. Rapoport, "Lewis F. Richardson's Mathematical Theory of War," J. Conflict Resolution 1, 249-299 (1957).

76. H. Rattinger, "Armaments, Detente, and Bureaucracy," J. Conflict Resolution 19, 571-595 (1975).

77. J. Record, Sizing Up the Soviet Army, The Brookings Institution, Washington, D.C., 1975.

78. L. F. Richardson, Arms and Insecurity, The Boxwood Press, Pittsburg and Quadrangle Books, Chicago, 1960.

79. L. F. Richardson, Statistics of Deadly Quarrels, The Boxwood Press, Pittsburg and Quadrangle Books, Chicago, 1960.

80. B. Rudwick, Systems Analysis for Effective Planning: Principles and Cases, John Wiley, New York, 1969.

81. T. Saaty, Mathematical Models of Arms Control and Disarmament, John Wiley. New York, 1968.

82. W. Schneider and F. Hoeber (Editors), Arms, Men, and Military Budgets, Crane, Russak and Company, New York, 1976.

83. M. Shubik, "Modeling on a Grand Scale," Science 174, 1014-1015 (1971).

84. M. Shubik, Games for Society, Business and War, Elsevier, New York, 1975.

85. M. Shubik, The Uses and Methods of Gaming, Elsevier, New York, 1975.

86. M. Shubik and G. Brewer, "Models, Simulations and Games - A Survey," R-1060-ARPA/RC, The RAND Corporation, Santa Monica, California, May 1972.

87. M. Shubik and G. Brewer, "Reviews of Selected Books and Articles on Gaming and Simulation," R-732-ARPA, The RAND Corporation, Santa Monica, California, June 1972.

88. M. Shubik, G. Brewer, and E. Savage, "The Literature of Gaming, Simulation, and Model-Building: Index and Critical Abstracts," R-620-ARPA, The RAND Corporation, Santa Monica, California, June 1972.

89. J. Stockfisch, Plowshares into Swords: Managing the American Defense Establishment, Mason and Lipscomb, New York, 1973.

90. J. Stockfisch, "Models, Data, and War: A Critique of the Study of Conventional Forces," R-1526-PR, The RAND Corporation, Santa Monica, California, March 1975.

91. J. Stockfisch, "Incentives and Information Quality in Defense Management," R-1827-ARPA, The RAND Corporation, Santa Monica, California, August 1976.

92. Studies, Analysis, and Gaming Agency, Organization of the Joint Chiefs of Staff, "Catalog of War Gaming and Military Simulation Models (Sixth Edition)," SAGA-236-75, Washington, D.C., June 1975.

93. J. Taylor and G. Brown, "Canonical Methods in the Solution of Variable-Coefficient Lanchester-Type Equations of Modern Warfare," Opns. Res. 24, 44-69 (1976).

94. J. Taylor and S. Parry, "Force-Ratio considerations for Some Lanchester-Type Models of Warfare," Opns. Res. 23, 422-533 (1975).

95. C. J. Thomas, "The Genesis and Practice of Operational Gaming," pp. 64-80 in Proc. First International Conference on Operational Research, M. Davies, R. Eddison, and T. Page (Editors), Operations Research Society of America, Baltimore, 1957.

96. C. J. Thomas, "Military Gaming," pp. 421-463 in Progress in Operations Research, Volume I, R. Ackoff (Editor), John Wiley, New York, 1961.

97. C. J. Thomas and W. Deemer, "The Role of Operational Gaming in Operations Research," Opns. Res. 5, 1-27 (1957).

98. L. Timothy and B. Bona, State Space Analysis: An Introduction, McGraw-Hill, New York, 1968.

99. TRW, "SIAF System Model, User's Manual, Volumes I-V," TRW Systems Group Redondo Beach, California, August 1971.

100. U.S. Army Air Defense School, "Deterministic Mix Evaluation World-wide (DMEW), Documentation of DMEW 1.4, Part I (Methodology)," Fort Bliss, Texas, September 1973.

101. U.S. Army Concepts Analysis Agency, "Tabulation of Models of Interest to CAA," Bethesda, Maryland, January 1977.

102. A. Vagts, "Battle and Other Combatant Casualties in the Second World War, I," J. Politics 7, 256-294 (1945).

103. A. Vagts, "Battle and Other Combatant Casualties in the Second World War, II," J. Politics 7, 411-438 (1945).

104. Vector Research, Inc., "Analytic Models of Air Cavalry Combat Operations," Report No. SAG-1 FR 73-1, Ann Arbor, Michigan, May 1973.

105. Vector Research, Inc., "VECTOR-0, The BATTLE Model Prototype," WSEG Report 222, Ann Arbor, Michigan, December 1973.

106. Vector Research, Inc., "DIVOPS: A Division-Level Combined Arms Engagement Model," Report No. ARAFCAS-1 TR-74, Vol. I-II, Ann Arbor, Michigan, 1974.

107. Vector Research, Inc.," VECTOR-1, A Theater Battle Model," WSEG Report 251, Vol. I-II, Ann Arbor, Michigan, July 1974.

108. H. M. Wagner, Principles of Operations Research, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.

109. Weapon System Effectiveness Industrial Advisory Committee (WSEIAC), Final Report of Task Group II, "Prediction-Measurement (Concepts, Task Analysis, Principles of Model Construction)," AFSC-TR-65-2 - Vol. II, Air Force Systems Command, Andrews Air Force Base, Maryland, January 1965.

110. H. K. Weiss, "Requirements for a Theory of Combat," Memorandum Report No. 667, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, April 1953 (AD 13 717).

111. H. K. Weiss, "Lanchester-Type Models of Warfare," pp. 82-98 in Proc. First International Conference on Operational Research, M. Davies, R. Eddison, and T. Page (Editors), Operations Research Society of America, Baltimore, 1957.

112. H. K. Weiss, "Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System," Opns. Res. 7, 180-196 (1959).

113. H. K. Weiss, "Arms Control," IRE Trans. on Human Factors in Electronics, Vol. HFE-3, 4-13 (1962).

114. H. K. Weiss, "Stochastic Models for the Duration and Magnitude of a 'Deadly Quarrel'," Opns. Res. 11, 101-121 (1963).

115. H. K. Weiss, "Combat Models and Historical Data: The U.S. Civil War," Opns. Res. 14, 759-790 (1966).

116. A. Wilson, The Bomb and the Computer, Dell Publishing Co., New York, 1970.

117. Q. Wright, A Study of War, The University of Chicago Press, Chicago, 1942.

118. J. P. Young, "A Survey of Historical Developments in War Games," ORO-SP-98, Operations Research Office, The Johns Hopkins University, Bethesda, Maryland, March 1959 (AD 210 865).

119. R. E. Zimmerman, "A Monte Carlo Model for Military Analysis," pp. 376-400 in Operations Research for Management, Volume II, J. McCloskey and J. Coppinger (Editors), Johns Hopkins Press, Baltimore, 1956.

120. R. E. Zimmerman, "Simulation of Tactical War Games," pp. 711-762 in Operations Research and Systems Engineering, C. Flagle, W. Higgins, and R. Roy (Editors), Johns Hopkins Press, Baltimore, 1960.

Chapter 2. LANCHESTER'S CLASSIC COMBAT FORMULATIONS

2.1 Lanchester's Original Work.

In 1914[1] F. W. LANCHESTER[2] [55] considered his now classic mathematical formulations of combat between two homogeneous forces in order to quantitatively justify the principle of concentration[3] under "modern conditions." When viewed in this light, his simple differential equation models are quite reasonable. With the elegance of simplicity, they convincingly show that concentration of forces is much more important under "modern conditions" than under "ancient conditions."

We should, perhaps, be more amazed that such simple models yield intuitively appealing results than be critical because of the factors omitted from them (see WEISS [98, p. 15]). As is usually the case with simple analytical models (see Section 1.3 above), they may be too abstract to solve any specific real operational problem. They can, however, illustrate a general principle such as concentration, clearly delineate modelling issues, warn about potential difficulties, and serve as a basis for communication among analysts (see SHUBIK and BREWER [74, pp. 2-3] for further discussion). In other words, such simple analytical models can provide valuable insights into the dynamics of combat, although they may be far too simple to be able to address any specific operational problem.[4]

LANCHESTER's [55] hypothesis was simply the following. In "ancient times," warfare was essentially a sequence of one-on-one duels[5] so that the casualty-exchange ratio during any period of battle did not depend on the combatants' force levels. But under "modern conditions," however, the firepower of weapons widely separated in firing location can be concentrated on surviving targets so that each side's casualty rate is proportional

to the number of enemy firers and the casualty-exchange ratio consequently depends inversely on the force ratio. Hence, under modern (i.e. 1914) conditions there is a definite advantage to be gained from concentration of forces; this has not always been true since in ancient times there was no such advantage to be usually gained from concentration[6]. LANCHESTER stressed that "modern" technology had radically changed the fundamental nature of warfare from what it was in the past. In ancient times, weapons such as swords and battle axes had to directly engage each other so that warfare was essentially a sequence of one-on-one duels. However, in modern times, the long-range delivery capability of contemporary weapons allows the concentration of firepower by weapons widely separated in firing location. Consequently, many weapons may fire at a few with devastating effects.

LANCHESTER's [55] main contribution was to translate the above verbal model[7] into mathematical terms. Because of the really pioneering nature of his work, LANCHESTER provided much motivation and logical (but not scientific) justification for his simple mathematical developments. He [55, p. 422] very insightfully comments that "the defense of modern times is indirect: tersely, the enemy is prevented from killing you by your killing him first, and the fighting is essentially collective." The model that LANCHESTER formulated for combat under modern conditions reflects this consideration. He then used this model to convincingly show the advantage from concentration of forces, i.e. the advantage of not committing forces "piecemeal."

<u>Conditions of Ancient Warfare</u>. As we have seen above, LANCHESTER hypothesized that ancient warfare was essentially composed of a series

of one-to-one duels between men fighting with weapons such as swords, battle axes, etc. He argued that if two equal-sized forces composed of combatants with equal fighting ability were to meet in battle, then each side would lose about the same number of men. Let us denote one side as the X force and the other as the Y force. Then LANCHESTER reasoned that if 1000 members of the X force and 1000 of the Y force meet in battle, it is of little consequence whether, for example, the 1000 X meet the entire Y force at once, or half now and the other half later. LANCHESTER reasoned (implicitly) that those who do not have duel opponents would have to wait in line for the opportunity to do battle and could not "gang up" on the enemy. In other words, there is no advantage to be gained from concentration of forces.

LANCHESTER did not give any equations for ancient warfare[8], but it is clear from reading his paper that he had in mind a combat attrition process for which the (instantaneous) casualty-exchange ratio is independent of the numbers of combatants, i.e.

$$\frac{dx}{dy} = E \quad , \tag{2.1.1}$$

where $x(t)$ and $y(t)$ denote the numbers of X and Y combatants at time t, and E denotes the constant exchange ratio. If we denote the initial number of X combatants at the beginning of battle at $t = 0$ as x_0, i.e. $x(0) = x_0$, and similarly for the Y force, then integration of (2.1.1) yields <u>LANCHESTER's linear law</u>

$$x_0 - x(t) = E\{y_0 - y(t)\} \quad . \tag{2.1.2}$$

The significant insight into the dynamics of combat, which the above simple analytical combat model readily yields, is that <u>under</u> such <u>ancient</u>

conditions of warfare there was no advantage to be gained from concentrating forces. We can see that this important result is an immediate consequence of LANCHESTER's linear law (2.1.2) by considering how a side's casualties depend on the number of his forces initially committed to battle[9]. Consider, for simplicity, a fight-to-the-finish in which the X force will be annihilated. [In Section 2.10 below, we will consider this topic again with more realistic battle-termination conditions after we have briefly considered the topic of modelling the battle-termination process.] Let us denote the final force levels at the end of battle with the subscript "f," and then $x_f = 0$. Let us also assume that the exchange ratio E is equal to unity, i.e. $E = 1$, and that X starts with 100 men, i.e. $x_0 = 100$. Then, we can take different values for Y's initial strength, use (2.1.2) to compute y_f, and determine Y's loss for each different initial commitment of forces. As Table 2.I shows us, we find that Y's loss is always the same (provided that Y wins, i.e. $y_0 \geq 100$), irrespective of how many men he commits to battle. Although we have demonstrated this result only for specific numerical values, it is true in general (see Section 2.10 below). Thus, there is no advantage under conditions of ancient warfare to concentrating forces.

Modern Conditions Investigated. LANCHESTER hypothesized that under "modern conditions," a side's casualty rate would be proportional to the number of enemy combatants due to the firepower-delivery capability of modern weapons. In mathematical terms, we have

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = x_0, \\[2ex] \dfrac{dy}{dt} = -bx & \text{with } y(0) = y_0, \end{cases} \tag{2.1.3}$$

TABLE 2.I Numerical Results That Illustrate That Under "Ancient Conditions" of Warfare There Was No Advantage to Concentrating Forces (i.e. No Reduction in Own Casualties From Committing More Men to Battle).

"ANCIENT WARFARE"

$$x_0 - x_f = E\ (y_0 - y_f)$$

Set $E = 1$, $x_0 = 100$, $x_f = 0$

Then

y_0	100	150	200	250	300	500
y_f	0	50	100	150	200	400
Y's loss	100	100	100	100	100	100

NO ADVANTAGE TO CONCENTRATING FORCES

where t denotes the battle time, the battle begins at $t = 0$, and a and b are constants that are today called LANCHESTER attrition-rate coefficients. These attrition-rate coefficients represent the effectiveness of each side's fire (i.e. its firepower). This simple combat situation considered by LANCHESTER is diagrammatically represented in Figure 2.1.

In contrast to the previous situation for ancient warfare, it now makes a tremendous difference how the Y force of 1000 combatants is committed against the X force of 1000 combatants. If all 1000 Y meet the 1000 X of equal fighting ability (i.e. we assume that the relative fire effectiveness, $\frac{a}{b}$, is equal to unity, namely $\frac{a}{b} = 1$), then the battle would be fought to a draw, with both sides being simultaneously annihilated. However, if half the Y force, i.e. 500 combatants, meets the entire X force, the result would be the annihilation of all the Y forces committed at a cost of about 134 casualties to X. Plots of the decays of the force levels are shown in Figure 2.2. If the 866 X survivors now engage the remaining 500 Y, the result would again be the annihilation of the Y combatnats, this time at a cost of about 159 additional casualties to X (see Figure 2.3). Thus, if X can divide the Y force and concentrate all his forces against each half in two sequential battles, then the entire Y force of 1000 men can be annihilated by X with a loss of only 293 men. LANCHESTER [55] gave this example and then went on to examine several other examples of the "weakness of a divided force." Thus, we see that under the "conditions of modern warfare" (at least as modelled by (2.1.3)) JULIUS CAESAR's famous dictum "divide and conquer" has been quantitatively justifed (at least in a heuristic sense).

From equations (2.1.3) we may obtain the instantaneous casualty-exchange ratio

LANCHESTER (1914)

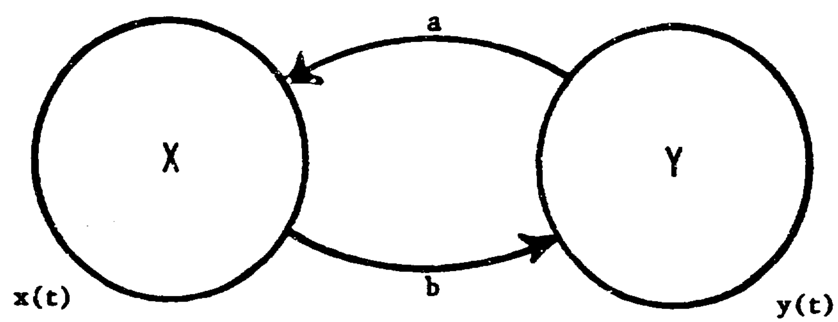


Figure 2.1. Diagram of simple combat situation considered by LANCHESTER.

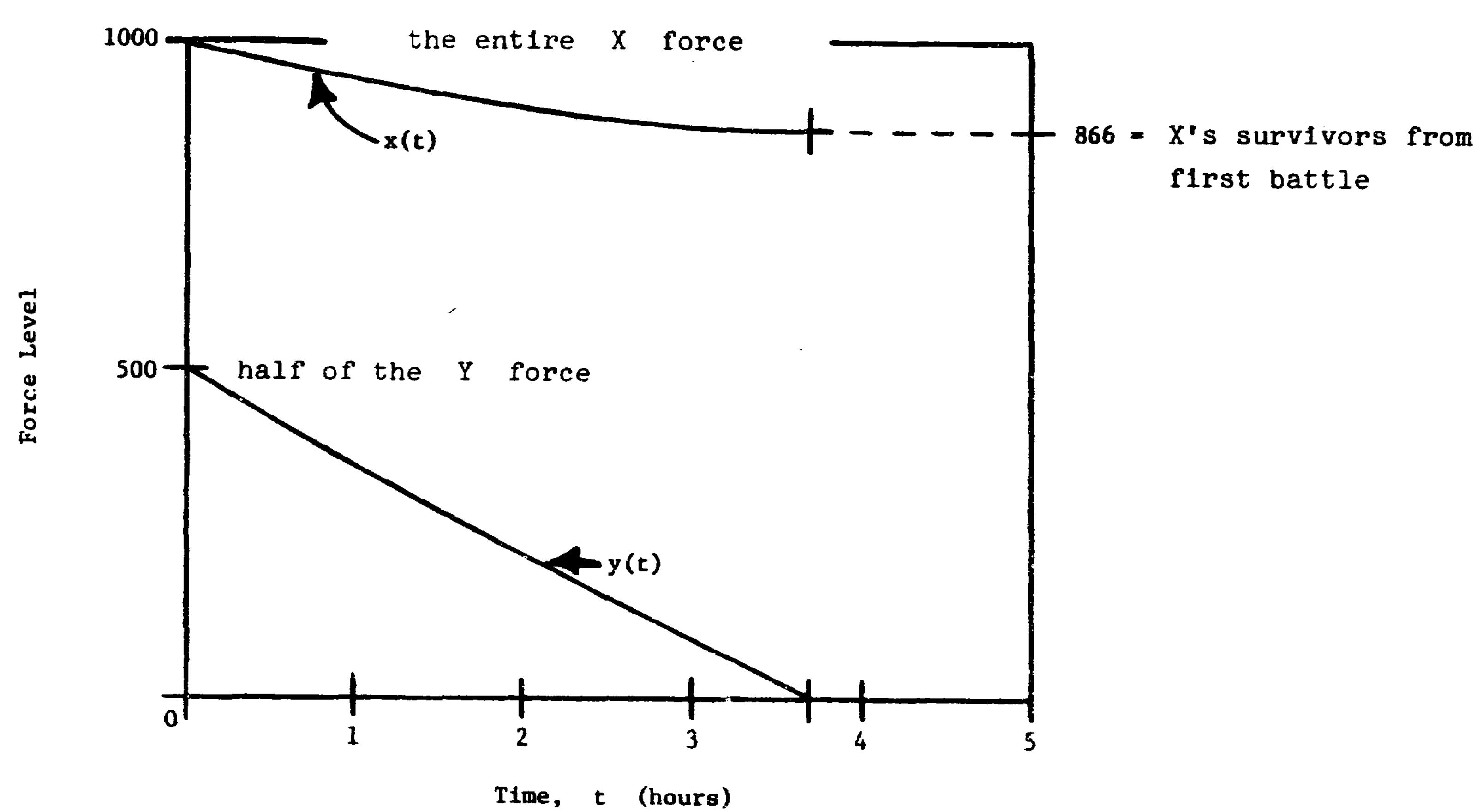


Figure 2.2. Decay in the force levels when half of the Y force meets in battle the entire X force of 1000 men.

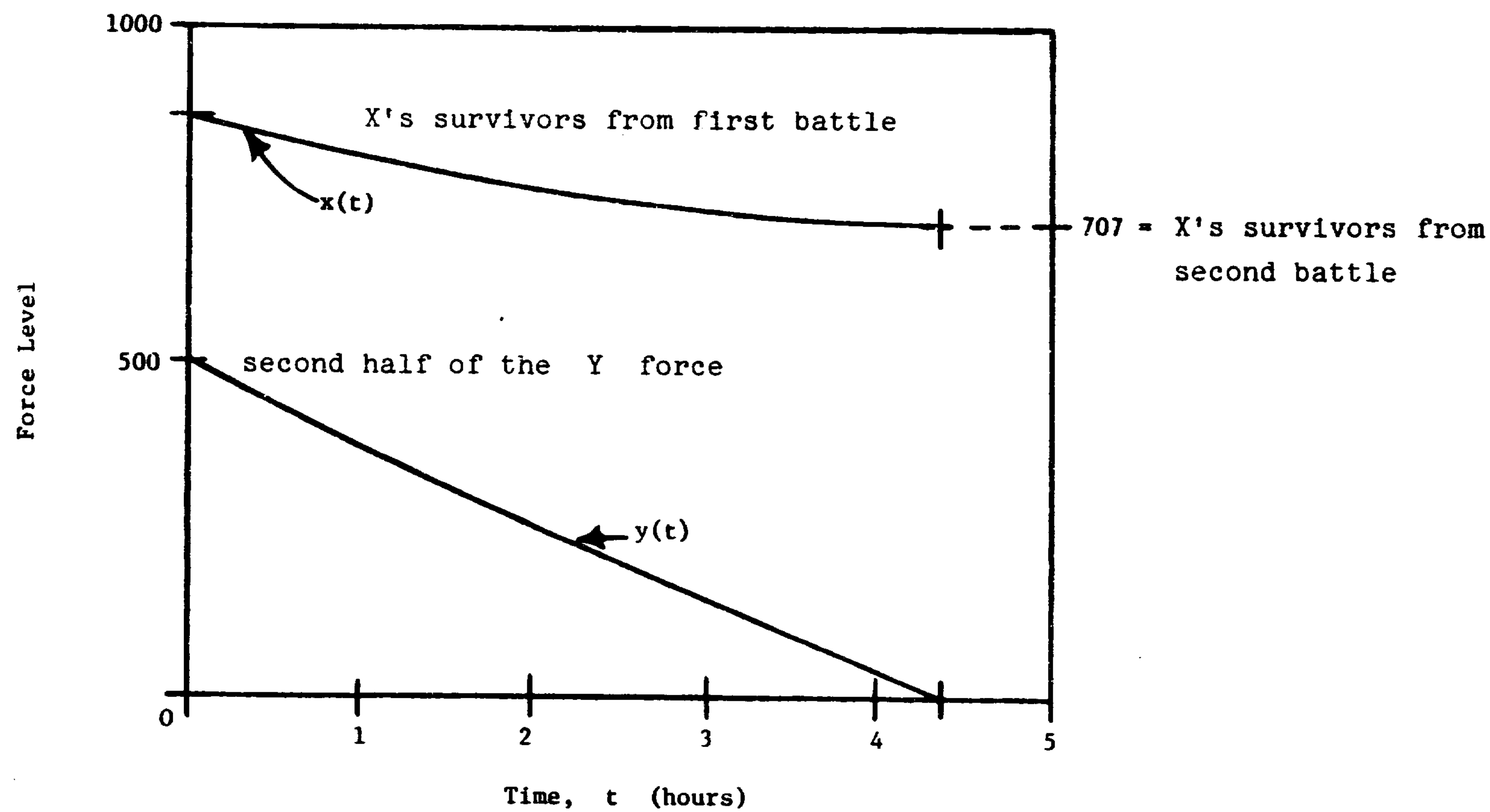


Figure 2.3. Decay in the force levels when the second half of the Y force meets the surviving X force in a second battle.

$$\frac{dx}{dy} = \frac{ay}{bx} = E\,\frac{y}{x} \quad , \tag{2.1.4}$$

where the constant exchange-ratio coefficient $E = a/b$ has been introduced so that we can readily compare (2.1.1) and (2.1.4). Integration of (2.1.4) yields LANCHESTER's square law

$$b(x_0^2 - x^2) = a(y_0^2 - y^2) \quad , \tag{2.1.5}$$

which (as we have partially seen above) has the important consequence that a side can significantly reduce its own casualties by initially committing more forces to battle (see Table 2.II and compare with Table 2.I).

LANCHESTER, however, referred to the "condition for equality of fighting strengths[10]," namely

$$bx^2 = ay^2 \tag{2.1.6}$$

as the "square law." It is interesting to note that he did not deduce (2.1.6) from (2.1.5),[11] but LANCHESTER [55, p. 422, column 1] reasoned that two forces are of equal strength when their force ratio does not change during the course of battle. For example, let an X force of 1000 combatants, each armed with an M-16, engage a Y force of 500 men, each armed with a light machine gun. If after a given time, X will have lost 200 men against a loss of 100 for Y, then the force ratio has remained constant and the forces may be regarded as being of equal strength. Introducing the force ratio, $u = x/y$, we find that it satisfies the RICCATI equation

$$\frac{du}{dt} = b\,u^2 - a \qquad \text{with } u(0) = u_0 = \frac{x_0}{y_0} \ . \tag{2.1.7}$$

From (2.1.7) we see that the force ratio doesn't change over time

TABLE 2.II Numerical Results That Illustrate That Under "Modern Conditions" of Warfare There Is an Advantage to Concentrating Forces (i.e. Reduction in Own Casualties From Committing More Men to Battle).

"MODERN WARFARE"

$$x_0^2 - x_f^2 = E\,(y_0^2 - y_f^2)$$

Set $E = 1$, $x_0 = 100$, $x_f = 0$

Then

y_0	100	150	200	250	300	500
y_f	0	112	173	229	283	490
Y's loss	100	38	27	21	17	10

ADVANTAGE TO CONCENTRATING FORCES

(i.e. $du/dt \equiv 0$) if and only if (2.1.6) holds. It was indeed insightful that LANCHESTER deduced his famous "square law" in this fashion.[12]

Area-Fire Model. LANCHESTER also considered the case in which each side fires into the general area occupied by the enemy and not at particular targets. He assumed that this area is independent of the number of targets present in the area. Implicit in LANCHESTER's development is the assumption that fire is uniformly distributed over this area. In this case, LANCHESTER hypothesized that the following equations would hold

$$\begin{cases} \dfrac{dx}{dt} = -\,axy & \text{with } x(0) = x_0, \\ \dfrac{dy}{dt} = -\,bxy & \text{with } y(0) = y_0. \end{cases} \tag{2.1.8}$$

Again, (2.1.2) is a consequence of (2.1.8) with $E = a/b$, so that in such cases of area-fire battles there is no particular advantage from concentration (again, see Table 2.I).

Final Remarks. The level of mathematics is kept at a minimum in LANCHESTER's original paper [55], yet if one carefully reads the paper, it becomes clear that LANCHESTER had explored fairly deeply the mathematical properties and operational implications of his simple models. In the next couple of sections we will examine the properties, behavior, and operational implications of these classic models.

2.2. Constant-Coefficient LANCHESTER-Type Equations for Modern Warfare.

We have seen that in his original 1914 paper, LANCHESTER [55] hypothesized that combat between two homogeneous forces under "modern conditions" could be modelled by[13]

$$
\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -bx & \text{with } y(0) = y_0 . \end{cases} \tag{2.2.1}
$$

Even though combat is a complex random process, such deterministic differential-equation models are commonly used in the analysis of military combat.[14] In this simple combat model, the attrition rate for each force, e.g. $(-dx/dt)$ for the X force, is assumed to be proportional to only the number of enemy firers. As we have seen above, the constants a and b represent the effectiveness of each side's fire, i.e. its firepower, and are called LANCHESTER attrition-rate coefficients. In other words, the attrition-rate coefficient a represents the fire effectiveness of a single Y firer, i.e. the rate at which he kills X targets.

This simple combat model is very significant because almost all developments in the LANCHESTER theory of combat [including current operational models such as BONDER/IUA, BLDM, VECTOR-2, etc. (see Section 1.3)] may in one sense or another be considered to take (2.2.1) as a point of departure. In particular, much can be learned about developing analytical solutions and gaining insights into the dynamics of combat by studying it. Consequently, we will study this particular model in some detail.

For convenience, we will refer to the equation (2.2.1) as LANCHESTER's equations for modern warfare,[15] although they have been hypothesized to apply under other circumstances. In fact, two sets of physical circumstances under which these equations have been hypothesized to apply are:

(C1) both sides use "aimed" fire and target-acquisition times are constant, independent of the force levels (a special case of which is when target acquisition times are negligible) [99],

(C2) both sides use "area" fire and a constant density defense [15].

A more complete discussion of these hypotheses is to be found in the papers by BRACKNEY [15] and WEISS [99] and in Section 2.11 below.

The above equations (2.2.1) only make sense for $x, y \geq 0$, since negative force levels are physically meaningless. If we consider the physical process of two military forces exchanging fire, then it is clear that equations (2.2.1) can only be valid for $x, y > 0$ and require modification for $x = 0$ or $y = 0$. For example, the first becomes $dx/dt = 0$ for $x = 0$. To be more precise, we should write LANCHESTER's classic model of modern warfare as

$$\left\{ \begin{aligned} \frac{dx}{dt} &= \begin{cases} -ay & \text{for } x > 0, \\ 0 & \text{for } x = 0, \end{cases} \\ \frac{dy}{dt} &= \begin{cases} -bx & \text{for } y > 0, \\ 0 & \text{for } y = 0. \end{cases} \end{aligned} \right. \tag{2.2.2}$$

To avoid inessential complications, however, we will not do so with the understanding that when we write the differential equations for some model like (2.2.1), we implicitly imply that the equations are "turned off" when, for example, one side or the other is annihilated. The reader should also observe from (2.2.2) that a LANCHESTER-type differential-equation combat model need not always have the same "right-hand sides."

The next aspect to consider is to determine what we can learn from LANCHESTER's model of modern warfare about the dynamics of combat between two homogeneous forces. In particular, one is interested in answering such questions[16] as:

(Q1) Who will "win"? Be annihilated?

(Q2) What force ratio is required to guarantee victory?

(Q3) How many survivors will the winner have?

(Q4) How long will the battle last?

(Q5) How do the force levels change over time?

(Q6) How do changes in parameters [i.e. initial force levels, x_0 and y_0, and attrition-rate coefficients, a and b] affect the outcome of battle?

(Q7) Is concentration of forces a good tactic?

In the remainder of this section we will consider answering the above questions.

The two basic vehicles for answering the above questions are (1) the state equation, and (2) the X (or Y) force level as a function of time. Additionally, we will see that we can also determine who will be annihilated from the force-ratio equation and obtain further insights into the dynamics of combat.

A state equation is an equation satisfied by the state variables. Since time t is not a state variable, the state equation for combat between two homogeneous forces takes the general form

$$S(x,y) = 0 , \tag{2.2.3}$$

where x and y denote the force levels of X and Y, respectively. To obtain the state equation for the combat model (2.2.1), we divide the first equation by the second to obtain the instantaneous (or differential) casualty-exchange ratio

$$\frac{dx}{dy} = \frac{ay}{bx} . \tag{2.2.4}$$

Separating variables and integrating, we obtain the state equation for LANCHESTER's model of modern warfare

$$b\{x_0^2 - x^2(t)\} = a\{y_0^2 - y^2(t)\} . \tag{2.2.5}$$

We will also refer to (2.2.5) as LANCHESTER's square law.

Let us now see how we may use the above state equation to obtain the X force level as a function of time, denoted as $x(t)$, for combat modelled by (2.2.1). Solving for x and substituting into the first differential equation of (2.2.1), we obtain

$$\frac{dx}{d\tau} = -\sqrt{x^2+k} \quad \text{with initial condition } x(\tau = 0) = x_0, \tag{2.2.6}$$

where $\tau = \sqrt{ab}\, t$ and $k = (a/b)y_0^2 - x_0^2$. Separating variables and integrating, we find that

$$\ln\left(\frac{x + \sqrt{x^2+k}}{x_0 + y_0\sqrt{a/b}}\right) = -\tau . \tag{2.2.7}$$

Raising e to the power of each side of (2.2.7), we obtain the X force level $x(t)$ after some algebraic manipulation

$$x(t) = \frac{1}{2}\left\{(x_0 - \sqrt{\tfrac{a}{b}}\, y_0)\, e^{\sqrt{ab}\, t} + (x_0 + \sqrt{\tfrac{a}{b}}\, y_0)\, e^{-\sqrt{ab}\, t}\right\} . \tag{2.2.8}$$

In terms of the so-called hyperbolic functions (see Appendix A.1), we may write the X force levels as

$$x(t) = x_0 \cosh \sqrt{ab}\, t - y_0\sqrt{\frac{a}{b}} \sinh \sqrt{ab}\, t \tag{2.2.9}$$

For the general case of time-dependent attrition-rate coefficients,[17] there is no state equation of the form $S(x,y) = 0$. With this fact in mind,

let us seek another method that does not depend on using such a state equation to develop the X force level. We may differentiate the first equation of (2.2.1) with respect to t and combine the result with the second equation to obtain a second order linear ordinary differential equation that contains only the X force level

$$\frac{d^2x}{dt^2} - abx = 0 , \qquad (2.2.10)$$

with initial conditions

$$x(0) = x_0 , \qquad \text{and} \qquad \frac{dx}{dt}(0) = -ay_0 .$$

We will call (2.2.10) the <u>X force-level equation.</u> Using standard solution methods (<u>see</u> Appendix A.2), we again obtain (2.2.8) [or, equivalently, (2.2.9)] for the X force level. Again, this solution approach of developing an X force-level equation is significant because it generalizes to cases of variable coefficients, whereas the approach based on the state equation in general does not.

In Figures 2.4 and 2.5 is plotted the decay of the X and Y force levels. For convenience, we record these results here as[18]

$$x(t) = x_0 \cosh\sqrt{ab}\, t - y_0\sqrt{\frac{a}{b}} \sinh \sqrt{ab}\, t ,$$

and

$$y(t) = y_0 \cosh\sqrt{ab}\, t - x_0\sqrt{\frac{b}{a}} \sinh \sqrt{ab}\, t . \qquad (2.2.11)$$

The force levels are most conveniently expressed in terms of the hyperbolic functions when parametric studies are desired. We will see below that representation of the force levels in terms of the exponential functions provides certain important insights. In Figure 2.4 the smaller force is seen to be annihilated, whereas in Figure 2.5 the larger force is annihilated. In both cases, we have "stopped" the battle as soon as one side or the other

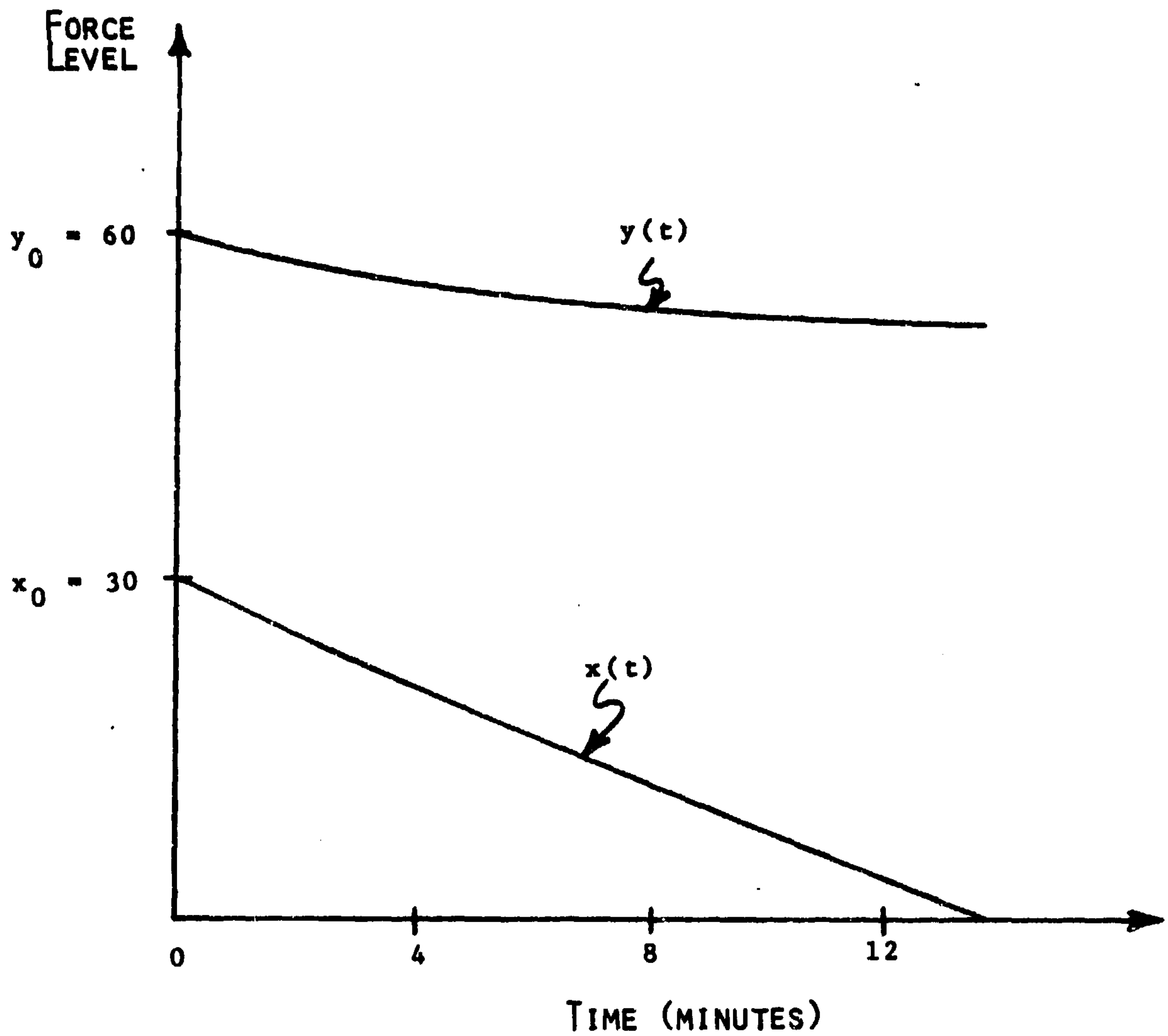


Figure 2.4. Force-level trajectories of X and Y forces for combat modelled by LANCHESTER's equations of modern warfare. For these calculations, $a = 0.04$ X casualties/(minute·number of Y combatants) and $b = 0.04$ Y casualties/(minute·number of X combatants).

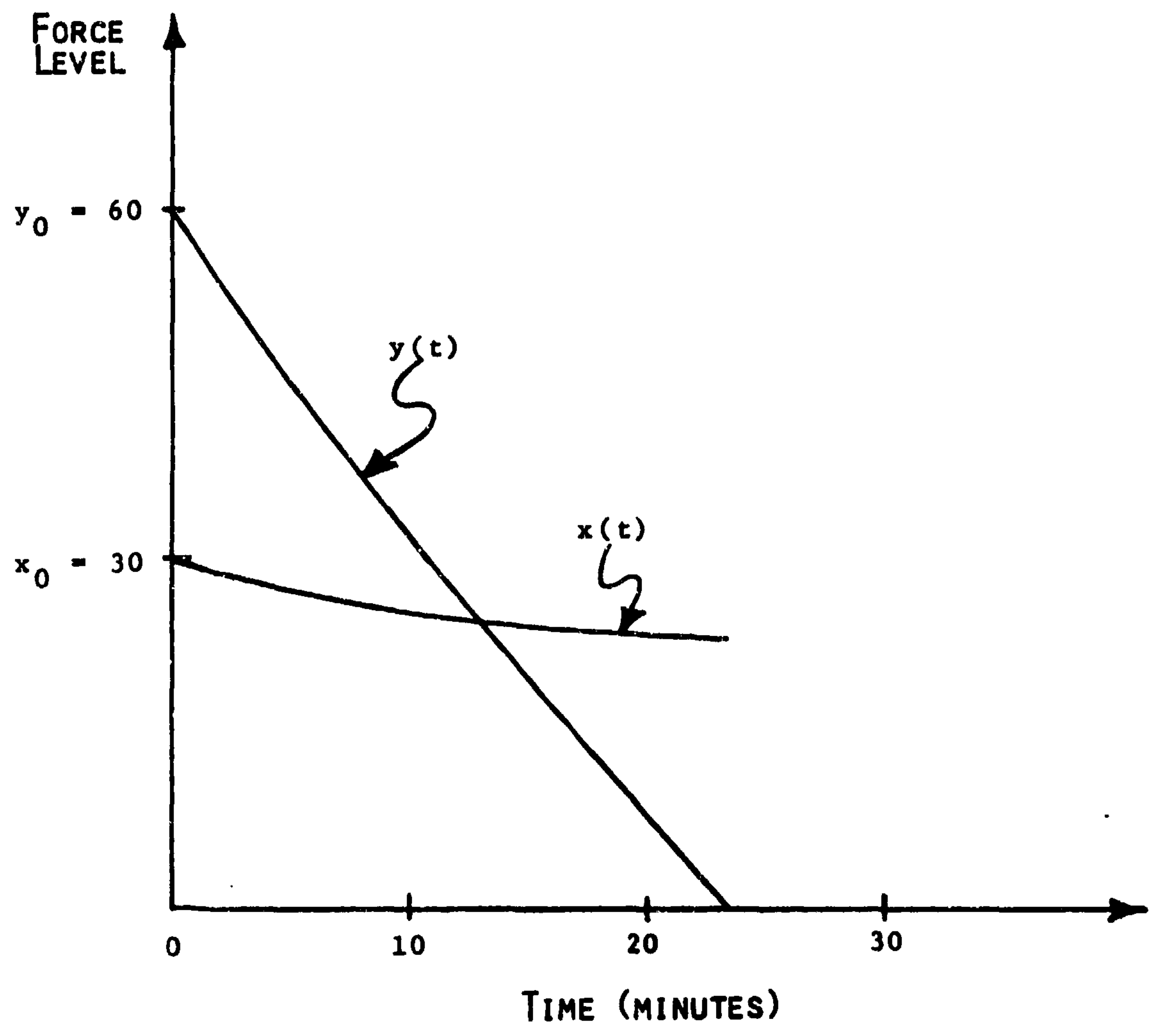


Figure 2.5. Force-level trajectories of X and Y forces for combat modelled by LANCHESTER's equations of modern warfare. For these calculations, $a = 0.01$ X casualties/(minute·number of Y combatants) and $b = 0.1$ Y casualties/(minute·number of X combatants).

has been annihilated, i.e. we have not computed the force levels past the time at which one side is first annihilated.

To more clearly exhibit the parametric dependence of the force-level trajectors, we normalize the force level by considering the fraction of the initial strength $x(t)/x_0$ given by

$$\frac{x(t)}{x_0} = \cosh \sqrt{ab}\ t - \frac{y_0}{x_0}\sqrt{\frac{a}{b}}\ \sinh \sqrt{ab}\ t \ . \qquad (2.2.12)$$

From (2.2.12) we see that the X force level depends on the following three quantities (although the model (2.2.1) contains the four independent parameters, a, b, x_0 and y_0):

(1) initial force ratio, $u_0 = x_0/y_0$,

(2) intensity of combat, $I = \sqrt{ab}$,

(3) relative fire effectiveness, $R = a/b$.

We observe that u_0 and R are relative quantities (without units), whereas I is an absolute quantity. It is the so-called geometric mean of the attrition-rate coefficients. It seems appropriate to call $I = \sqrt{ab}$ the intensity of combat, since the course of combat for the model (2.2.1) more quickly reaches its conclusion the larger that I is. In other words, I controls the time scale of battle.

To determine who will "win" the battle, one must specify battle-termination conditions, with "victory" conditions also being given for each side. In other words, one must have a model for the battle-termination process. The simplest, but albeit somewhat unrealistic in the light of historical evidence, model of battle termination is to consider that each side fights until it is annihilated. Let us assume that this is true. We will consider a more realistic model below in Section 2.8.

Thus, we consider a "fight-to-the-finish," which can have three possible outcomes:

(XW) X wins with $x_f > 0$ and $y_f = 0$,

(YW) Y wins with $y_f > 0$ and $x_f = 0$,

(D) draw with $x_f = y_f = 0$,

where x_f denotes the final X force level and similarly for y_f. For any particular battle (i.e. for particular specified values of the attrition-rate coefficients a and b and the initial force levels x_0 and y_0) we can <u>always</u> plot the decay of the force levels $x(t)$ and $y(t)$ versus time t and consequently determine who will be annihilated and who will win the fight-to-the-finish (<u>see</u> Figures 2.4 and 2.5). This is, however, a time-consuming procedure, and doesn't provide any deep understanding of the dynamics of combat, i.e. how weapon-system capabilities (as quantified by the attrition-rate coefficients a and b) and the initial force levels x_0 and y_0 determine the outcome of battle. However, it is of considerable interest to determine force-annihilation-prediction conditions, i.e. conditions that allow us to determine battle outcome (here, force annihilation) without having to spend the time and effort of explicitly computing force-level trajectories. Let us, therefore, now determine conditions that are necessary and sufficient for Y to win a fight-to-the-finish in finite time, i.e. X be annihilated in finite time. There are several ways in which we can do this. Here we will only consider the easiest way, with a more in depth examination being given in the next section.

Probably the easiest way to determine force-annihilation-prediction conditions is to consider the X and Y force levels expressed in terms of the exponential functions, namely

$$x(t) = \frac{1}{2}\{(x_0 - \sqrt{\frac{a}{b}}\, y_0)\, e^{\sqrt{ab}\, t} + (x_0 + \sqrt{\frac{a}{b}}\, y_0)\, e^{-\sqrt{ab}\, t}\} , \qquad (2.2.13)$$

and

$$y(t) = \frac{1}{2}\{(y_0 - \sqrt{\frac{b}{a}}\, x_0)\, e^{\sqrt{ab}\, t} + (y_0 + \sqrt{\frac{b}{a}}\, x_0)\, e^{-\sqrt{ab}\, t}\} . \qquad (2.2.14)$$

We observe that the second term in brackets for both $x(t)$ and $y(t)$ is always positive, since the negative exponential function is always positive. It is strictly decreasing as a function of t and becomes negligible for large t. Thus, both $x(t)$ and $y(t) > 0$ and $\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0$ if and only if $x_0/y_0 = \sqrt{a/b}$. In other words, we have a draw when (and only when) $x_0/y_0 = \sqrt{a/b}$. Furthermore, $y(t) > 0$ for all $t \geq 0$ and $\lim_{t \to +\infty} y(t) > 0$ if and only if the first term in brackets for (2.2.14) is positive, i.e. the coefficient of the increasing exponential in (2.2.14) is positive. This is equivalent to $x_0/y_0 < \sqrt{a/b}$. In this case (i.e. $x_0/y_0 < \sqrt{a/b}$) the first term in brackets of (2.2.13) for $x(t)$ is negative and decreases without bound as $t \to +\infty$. Hence, at some point in time $x = 0$ when the two terms in brackets just cancel out. Thus, we have shown

PROPOSITION 2.2.1: Y will win a fight-to-the-finish in finite time if and only if $x_0/y_0 < \sqrt{a/b}$.

Proposition 2.2.1 is particularly significant because it shows us that the outcome of battle (here, the annihilation of one side) is determined by only two relative factors (namely: (I) the initial force ratio $u_0 = x_0/y_0$, and (II) relative fire effectiveness, $R = a/b$) and not absolute

quantities. Thus, even though the model (2.2.1) contains the four independent parameters, it is the only two <u>relative</u> quantities u_0 and R that determine force annihilation. It is also very important for us to point out that (except for the so-called quasi-autonomous case in which $a(t)/b(t)$ is constant) although LANCHESTER's square law in the form (2.2.5) does not generalize to cases of time-dependent attrition-rate coefficients, the force-annihilation-prediction condition of Proposition 2.2.1 does generalize to such cases.

Rewriting (2.2.14) as

$$y(t) = \frac{1}{2}\sqrt{\frac{b}{a}}\{-(x_0 - y_0\sqrt{\frac{a}{b}})\, e^{\sqrt{ab}\, t} + (x_0 + y_0\sqrt{\frac{a}{b}})\, e^{-\sqrt{ab}\, t}\}\ , \quad (2.2.15)$$

we clearly see from (2.2.13) and (2.2.15) that at most one of X and Y can ever be annihilated in finite time (i.e. at most one of $x(t)$ and $y(t)$ can ever be driven to zero in finite time). This is an important property of the model (2.2.1), since it allows us to consider only one of $x(t)$ and $y(t)$ in order to determine force annihilation for both combatants. In other words, if $x(t_a^X) = 0$ with $t_a^X > 0$ and finite, then we know that $y(t) > 0$ for all $t \geq 0$. Thus, if we can compute the time for X to be annihilated, we know that $y(t)$ will always be greater than zero. In more mathematical terminology, equivalently, we have shown that the X force-level equation (2.2.10) possesses a <u>nonoscilliatory solution</u> $x(t)$, i.e. $x(t)$ has at most one zero for $t \in [0, +\infty)$. Furthermore, the same is true for $(dx/dt)(t)$.

In view of the importance of the fact that at most one of $x(t)$ and $y(t)$ is ever equal to zero, let us deduce this property of the

force level trajectories from the basic differential equations themselves. First, a few heuristics. Looking at the first equation of (2.2.1), we see that if $y(t)$ becomes negative, then $x(t)$ begins to increase. Thus, it is intuitively obvious that if $y(t)$ goes to zero and then becomes negative, the corresponding plot of $x(t)$ versus t will have a positive minimum corresponding to the time t_a^Y at which $y(t) = 0$. This situation is shown in Figure 2.6. Thus, if we forget to "turn off" equations (2.2.1) at t_a^Y (i.e. don't use (2.2.2)), then the X force level will actually increase as time t increases when $t > t_a^Y$.

Let us now give an analytical demonstration of the fact that all the solutions to (2.2.1) are nonoscillatory (<u>see</u> HILLE [38, p. 373]), i.e. at most one of $x(t)$ and $y(t)$ can vanish in finite time. Multiplying the first equation of (2.2.1) by y, the second by x, adding, and integrating the result between 0 and t, we obtain

$$x(t)\, y(t) = x_0 y_0 - \int_0^t \{ay^2(s) + bx^2(s)\}\, ds\ . \tag{2.2.16}$$

It is impossible for both $x(t)$ and $y(t)$ to be equal to zero at any finite time, since then they would have to be equal to zero for all time.[19] Hence, the integral term (i.e. $\int_0^t \{ay^2(s) + bx^2(s)\}ds$) is strictly increasing and positive for $t > 0$. Since $x_0 y_0 > 0$, it follows that $x(t)\, y(t)$ has at most one finite zero for $t \geq 0$. Thus, we have deduced the desired property, which we record here as Proposition 2.2.2.

PROPOSITION 2.2.2: For the model (2.2.1), at most one of the two force levels $x(t)$ and $y(t)$ can ever vanish in finite time.

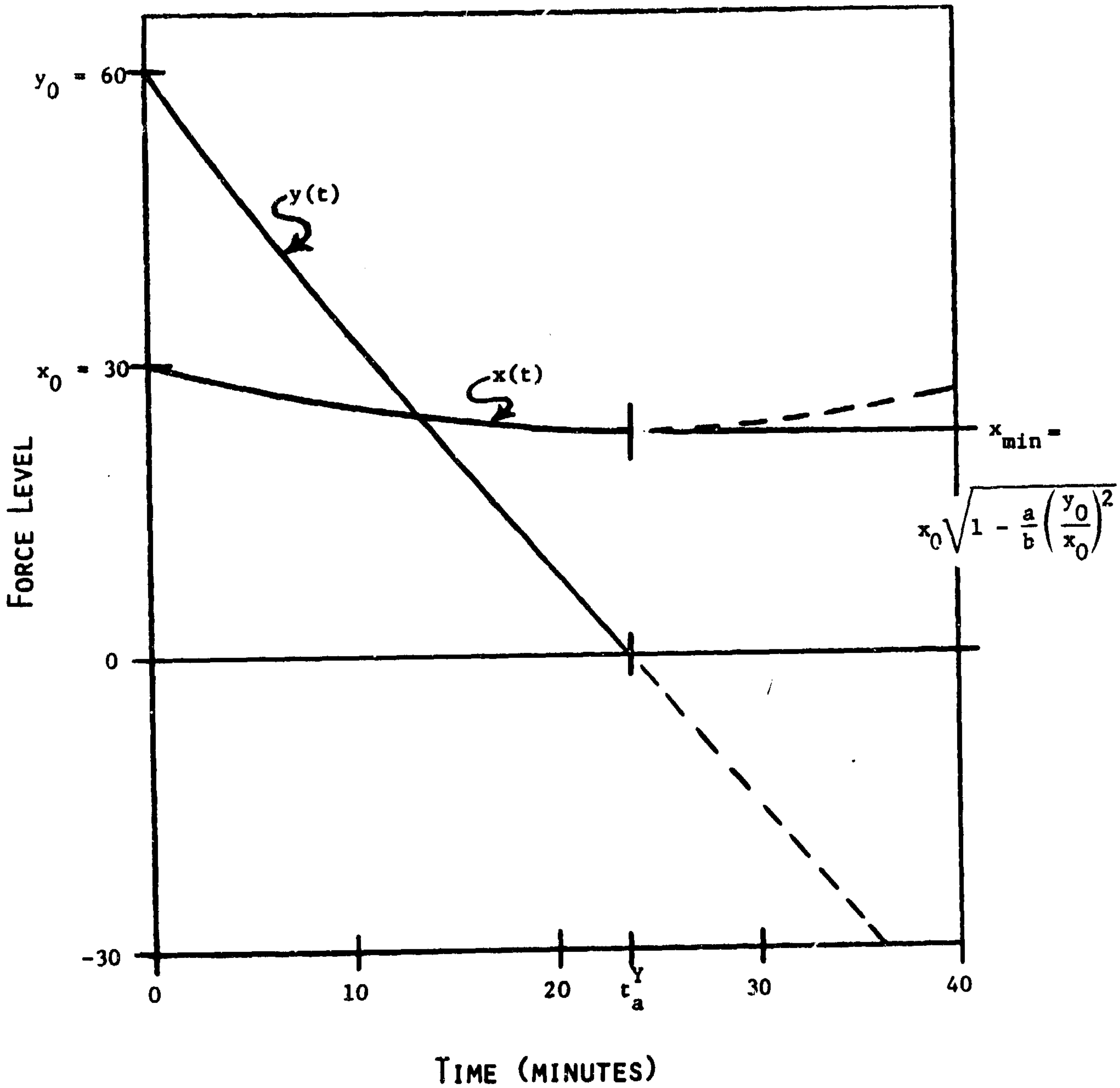


Figure 2.6. Force-level trajectories for combat modelled by the differential equations (2.2.1). The dashed lines extend the X and Y force levels computed by (2.2.13) and (2.2.14) past t_a^Y. The values of a and b are the same as for Figure 2.5.

Since the force-annihilation-prediction condition contained in Proposition 2.2.1 involves the initial force ratio and not a force level, we are motivated to consider the force ratio and ask what happens to it over the course of battle. Furthermore, many aggregated combat models (such as ATLAS) have both casualty rates and also FEBA movement depend on the force ratio (of firepower indices or their equivalent). In order to determine how the force ratio changes over time, we seek a differential equation for it. Introducing the force ratio $u = x/y$, we consequently consider $\ell n\, u = \ell n\, x - \ell n\, y$ and differentiate with respect to time to obtain

$$\frac{1}{u}\frac{du}{dt} = \frac{1}{x}\frac{dx}{dt} - \frac{1}{y}\frac{dy}{dt} .$$

Using the differential equations (2.2.1), we find that for the combat dynamics of (2.2.1) the force ratio $u = x/y$ satisfies the following Riccati equation (<u>see</u> Appendix A.3)

$$\frac{du}{dt} = bu^2 - a , \tag{2.2.17}$$

with $u(0) = u_0 = x_0/y_0$.

Although we could separate variables in (2.2.17) and integrate (<u>see</u> INCE [41, pp. 311-312]) to obtain[20]

$$u(t) = -\sqrt{\frac{a}{b}} \left\{ \frac{(x_0 - y_0\sqrt{a/b}) + (x_0 + y_0\sqrt{a/b})\ e^{-2\sqrt{ab}\ t}}{(x_0 - y_0\sqrt{a/b}) - (x_0 + y_0\sqrt{a/b})\ e^{-2\sqrt{ab}\ t}} \right\}, \tag{2.2.18}$$

the main use of the <u>force-ratio equation</u> (2.2.17) is not to solve explicitly for $u(t)$ but to obtain qualitative information about the solution $u(t)$. For a fight-to-the-finish, we observe that (a) X wins at $t = T$ when $u(T) = +\infty$, and (b) Y wins when $u(T) = 0$. Thus, it seems appropriate

to say that "the course of battle is moving towards a Y victory" when $du/dt < 0$ (or, simply, that "Y is winning"). Moreover, $du/dt < 0$ if and only if

$$\frac{x}{y} < \sqrt{\frac{a}{b}} \ . \tag{2.2.19}$$

Let us now examine the qualitative behavior of the force ratio over time as determined by the force-ratio equation (2.2.17). We will see that we need not solve (2.2.17), i.e. consider (2.2.18), in order to qualitatively determine how $u(t)$ changes over time. It seems appropriate to call du/dt the <u>force-ratio velocity</u>. For convenience we consider that (2.2.17) holds for $-\infty \leq u \leq +\infty$. Let us now examine how the force-ratio velocity du/dt depends on the force ratio u. For such an examination we hold t constant and consider du/dt to be a function of only u, denoted as $du/dt(u)$. We define $u_+ = \sqrt{a/b}$ and $u_- = -\sqrt{a/b}$. It follows from (2.2.17) that $du/dt(u) < 0$ for $u_- < u < u_+$. The minimum of $du/dt(u)$ occurs at $u_{min} = 0$, and we have $du/dt(u_{min}) = -a < 0$. Usually, however, we will let t vary, and then du/dt may be considered to be a function of t, denoted as $du/dt(t)$, since the dependent variable u depends on t.

In Figure 2.7 the force-ratio velocity du/dt is plotted against the force ratio u. It should be recalled that a negative force-ratio velocity has the interpretation that Y is "winning" the battle (2.2.1). Also shown by means of arrows drawn along the u axis in Figure 2.7 is the direction of movement of the force ratio, with the length of the arrow reflecting the magnitude of the force-ratio velocity. From Figure 2.7 it is clear that if $du/dt\ (t = 0) < 0$, then $u(t)$ decreases and $du/dt(t)$ becomes more negative (as long as $u \geq 0$). Thus, we have proved

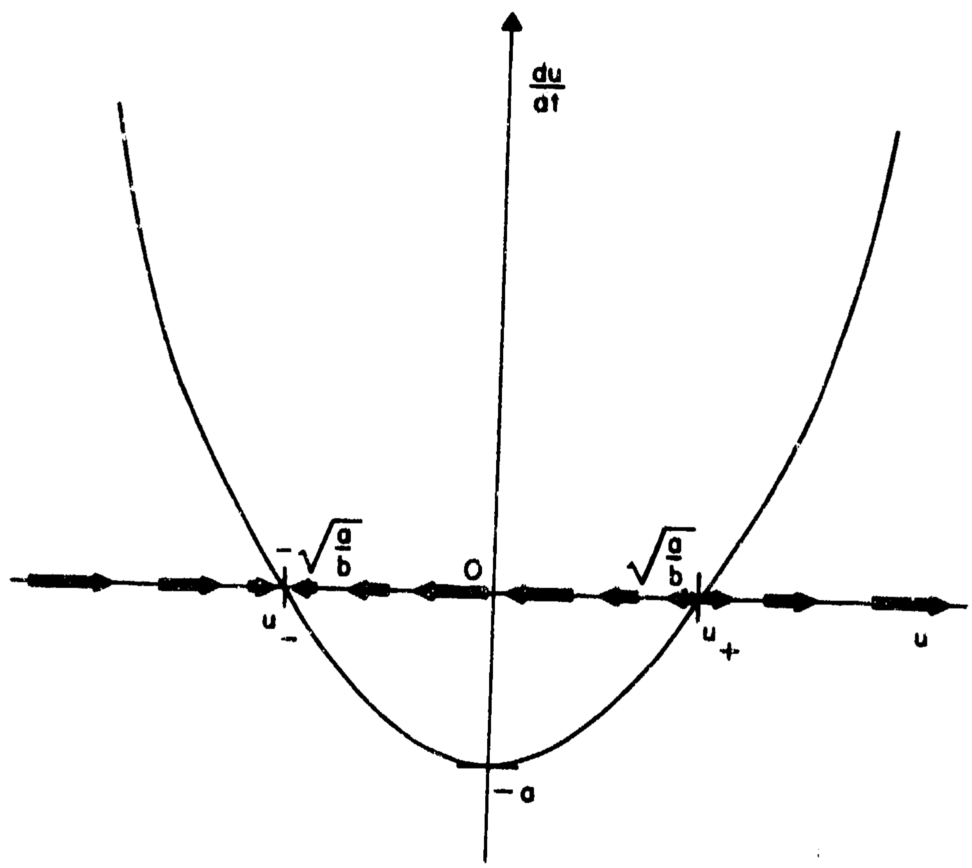


Figure 2.7. Force-ratio velocity as a function of the force ratio for combat modelled by LANCHESTER's equations of modern warfare.

PROPOSITION 2.2.3: If $du/dt(t = 0) < 0$, then $du/dt(t) < 0$ for all $t \geq 0$. If $u \geq 0$, then $du/dt(t) \leq du/dt(t = 0) < 0$.

Thus, if $u_0 < \sqrt{a/b}$, the force-ratio always will decrease during the course of battle; it will always increase if $u_0 > \sqrt{a/b}$. For the constant-coefficient model of "modern warfare" (2.2.1), we see from (2.2.17) that if $x_0/y_0 = \sqrt{a/b}$, then the force ratio remains constant during the course of battle although the force levels exponentially decline. We state this result as Proposition 2.2.4.

PROPOSITION 2.2.4: If $du/dt(t = 0) = 0$ (i.e. $x_0/y_0 = \sqrt{a/b}$), then the force ratio remains constant during the course of battle (i.e. $u(t) = x(t)/y(t) = \sqrt{a/b}$), although the force levels exponentially decrease, i.e. $x(t) = x_0 \exp(-\sqrt{ab}\, t)$ and $y(t) = y_0 \exp(-\sqrt{ab}\, t)$.

We observe that such force-level behavior only holds for a constant-coefficient[21] model.

Let us now show how the force-annihilation-prediction condition of Proposition 2.2.1 may be deduced from the force-ratio equation (2.2.17). This result is particularly significant because it generalizes to certain cases of time-dependent attrition-rate coefficients and yields simple force-annihilation-prediction results that do not involve any higher transcendental functions. We observe that $du/dt(t = 0) < 0$ if and only if $x_0/y_0 < \sqrt{a/b}$. Thus, by Proposition 2.2.3 $u(t)$ is strictly decreasing. It remains to show that $u(t)$ becomes zero in finite time. We readily show this by considering for $u \geq 0$

$$u(t) = u_0 + \int_0^t \left(\frac{du}{dt}\right) dt \leq u_0 + t \cdot \frac{du}{dt}(0) ,$$

the last inequality holding by Proposition 2.2.3. Hence, $u(t) \to 0$ in finite time, since $du/dt(0) < 0$.

We are now in a position to easily answer the question of how long the battle will last. Again, the results given here will be limited to a fight-to-the-finish. By proposition 2.2.1 we know that X will be annihilated if and only if $x_0/y_0 < \sqrt{a/b}$. The time at which X is annihilated, denoted as t_a^X, may be determined from $x(t_a^X) = 0$. In this determination we may express the X force level in terms of either the exponential functions [<u>see</u> equation (2.2.8)] or the hyperbolic functions [<u>see</u> equation (2.2.9)]. Thus, we have

$$t_a^X = \frac{1}{2\sqrt{ab}} \ln \left\{ \frac{1 + (x_0/y_0)\sqrt{b/a}}{1 - (x_0/y_0)\sqrt{b/a}} \right\} , \qquad (2.2.20)$$

or, equivalently,

$$t_a^X = \frac{1}{\sqrt{ab}} \tanh^{-1}\left(\frac{x_0}{y_0} \sqrt{\frac{b}{a}} \right) . \qquad (2.2.21)$$

The number of survivors for the winner (here Y) of this fight-to-the-finish may be determined by substituting the annihilation time t_a^X given by (2.2.20) into (2.2.14). Doing this, we obtain for the fractional survivors

$$\frac{y_f}{y_0} = \sqrt{1 - \frac{b}{a} \left(\frac{x_0}{y_0} \right)^2} , \qquad (2.2.22)$$

where y_f denotes the final Y force level at $t = t_a^X$. We also could have deduced (2.2.22) from LANCHESTER's square law (2.2.5) (i.e. the state equation for LANCHESTER's model of modern warfare) by setting $x(t) = x_f = 0$ and $y(t) = y_f$. We observe that the state equation (2.2.5) is useful for such determinations only when we already know one of the force levels.

In general, for $x, y \geq 0$ we have

$$\frac{y}{y_0} = \sqrt{1 - \frac{b}{a}\left\{\left(\frac{x_0}{y_0}\right)^2 - \left(\frac{x}{y_0}\right)^2\right\}} \quad . \tag{2.2.23}$$

The principal results that we have developed above are summarized in Table 2.III.

TABLE 2.III. Summary of Principal Results for LANCHESTER's Model of Modern Warfare

LANCHESTER's Equations for Modern Warfare

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with} \quad x(0) = x_0 \\[2ex] \dfrac{dy}{dt} = -bx & \text{with} \quad y(0) = y_0 \end{cases}$$

Differential Casualty-Exchange Ratio, $\frac{dx}{dy}$: $\qquad \frac{dx}{dy} = \frac{ay}{bx}$

State Equation: $\quad a\{y_0^2 - y^2(t)\} = b\{x_0^2 - x^2(t)\}$

Differential Equation Satisfied by the X Force Level:

$$\frac{d^2x}{dt^2} - abx = 0$$

with initial conditions

$$x(0) = x_0 \qquad \text{and} \qquad \frac{dx}{dt}(0) = -ay_0$$

X Force Level:

$$x(t) = x_0 \cosh\sqrt{ab}\, t - y_0 \sqrt{\frac{a}{b}} \sinh\sqrt{ab}\, t$$

or

$$x(t) = \frac{1}{2}\left\{\left(x_0 - y_0\sqrt{\frac{a}{b}}\right) e^{\sqrt{ab}\, t} + \left(x_0 + y_0\sqrt{\frac{a}{b}}\right) e^{-\sqrt{ab}\, t}\right\}$$

Differential Equation Satisfied by the Force Ratio, $u = \frac{x}{y}$:

$$\frac{du}{dt} = bu^2 - a \qquad \text{with} \quad u(0) = \frac{x_0}{y_0}$$

Force-Annihilation-Prediction Condition: X will be annihilated in finite time if and only if $x_0/y_0 < \sqrt{a/b}$.

*2.3. A Further Look at Predicting Force Annihilation.

It is important for the military operations analyst to have a clear understanding of how force-level and weapon-system-performance factors interact to determine the outcome of battle. Victory-prediction conditions (i.e. conditions that predict the outcome of battle without requiring the expenditure of time and effort to explicitly compute the force-level trajectories) provide important insights into the dynamics of combat by explicitly relating the initial force ratio and weapon-system capabilities to the outcome of battle. Consequently, we will examine in greater depth here the development of force-annihilation-prediction conditions for LANCHESTER's (constant-coefficient) equations for modern warfare (2.2.1). Our reasons for doing this are twofold:

(R1) to extend such victory-prediction conditions to other models [particularly the variable-coefficient version of (2.2.1)],

(R2) to develop other types of outcome-prediction conditions [e.g. victory-prediction conditions for a fixed-force-level-breakpoint battle (see Section 2.8 below)].

In other words, examining the various approaches for developing force-annihilation-prediction conditions provides us with important clues for extending such conditions to other cases of interest.

In Table 2.IV we list the six different approaches for developing force-annihilation-prediction conditions. For the combat model (2.2.1), the force-annihilation-prediction condition is given by Proposition 2.2.1,

---

*Starred sections are not required for the understanding of the sequel and should be omitted at first reading. They usually require more mathematical sophistication to be understood.

which we restate here for convenience.

PROPOSITION 2.3.1: Y will win a fight-to-the-finish in finite time if and only if $\frac{x_0}{y_0} < \sqrt{\frac{a}{b}}$.

The list given in Table 2.IV is exhaustive, i.e. we do not know of any other way to develop conditions that predict force annihilation. Moreover, it is approach (3b), determining the time to annihilation with the force levels represented in terms of the hyperbolic functions, that provides a computational means for determining force-annihilation-prediction conditions for the general case of time-dependent attrition-rate coefficients for the model (2.2.1). All the other approaches are not capable of being generalized to such cases of variable coefficients.

In all but the next to last approach (4), manipulation of the state equation (2.2.5), we will ultimately discover the nonoscillation of all solutions to (2.2.1), i.e. at most one of the force levels $x(t)$ and $y(t)$ can ever become zero (see Proposition 2.2.2). Since its proof does not depend on force-annihilation determination, let us assume that this important property of all solutions to (2.2.1) has been established. Knowledge of the existence of the nonoscillation property simplifies the development of force-annihilation-prediction conditions. Let us note, however, that this important nonoscillation property no longer generally holds when continuous replacements and/or withdrawals are added to the model (2.2.1).

Since approaches (1) and (2) of Table 2.IV have been considered in Section 2.2 above, we will not consider them further here except for making a few additional comments. First, we observe that analysis of the

TABLE 2.IV Approaches for Developing Force-Annihilation-Prediction Conditions

THESE APPROACHES ARE TO CONSIDER:

(1) X force level represented in terms of exponential functions

(2) force-ratio equation

(3) time to annihilation with force levels represented in terms of

  (a) exponential functions

  (b) hyperbolic functions

(4) state equation

(5) HELMBOLD's monotonicity condition (Method B of Section 3.3)

force-ratio equation (2.2.16) (see Figure 2.5) leads to another proof of the nonoscillation of all solutions to (2.2.1). Secondly, by both approaches (1) and (2), we readily establish that annihilation occurs in finite time (except for the case of a draw).

Approach (3a) consists of considering the X force level expressed in terms of the exponential functions [see equation (2.2.8)] and solving for the time for the X force to be annihilated, denoted as t_a^X, as determined by the equation $x(t_a^X) = 0$. Consequently, we find that

$$t_a^X = \frac{1}{2\sqrt{ab}} \ln \left\{ \frac{y_0\sqrt{a} + x_0\sqrt{b}}{y_0\sqrt{a} - x_0\sqrt{b}} \right\} . \tag{2.3.1}$$

In order for t_a^X to be well defined and positive, the argument of the logarithm must be greater than one (but finite), and hence $x_0/y_0 > a/b$ in order for X to be annihilated. By the nonoscillation of all solutions to (2.2.1) (i.e. Proposition 2.2.2), we know that $y(t) > 0$ for all $t \geq 0$ if there exists a finite t_a^X such that $x(t_a^X) = 0$, whence follows Proposition 2.3.1. We also observe that the nonoscillation of all solutions to (2.2.1) may also be proven by observing that

$$t_a^Y = \frac{1}{2\sqrt{ab}} \ln \left\{ \frac{x_0\sqrt{b} + y_0\sqrt{a}}{x_0\sqrt{b} - y_0\sqrt{a}} \right\} \tag{2.3.2}$$

and comparing this result with (2.3.1).

Approach (3b) consists of considering the X force level expressed in terms of the hyperbolic functions [see equation (2.2.8)] and again determining t_a^X from $x(t_a^X) = 0$. Hence,

$$t_a^X = \frac{1}{\sqrt{ab}} \tanh^{-1} \left(\frac{x_0}{y_0} \sqrt{\frac{b}{a}} \right) \tag{2.3.3}$$

Proposition 2.3.1 follows by observing that the hyperbolic tangent, i.e.

$\tanh \xi$, is a strictly increasing function with range [0,1] corresponding to $\xi \epsilon [0, +\infty]$. It is this property of the hyperbolic tangent that may be generalized to cases of time-dependent attrition-rate coefficients in order to develop the sought force-annihilation-prediction conditions.

Approach (4) consists of considering the state equation (2.2.5) and setting $x = x_f = 0$ and $y = y_f > 0$ to obtain

$$y_f^2 = y_0^2 - \frac{b}{a} x_0^2 > 0 \quad ,$$

which means that we must have $x_0/y_0 < \sqrt{a/b}$ in order that X will be annihilated. Thus, we have shown that $x_0/y_0 < \sqrt{a/b}$ is a <u>necessary</u> condition for the X force to be annihilated. However, to show that this condition is also sufficient is much more difficult. Even if we assume that Proposition 2.2.2 has been proven, it is still not a trivial task to show that the condition $x_0/y_0 < \sqrt{a/b}$ is sufficient to guarantee that X will be annihilated (and much less that it will occur in finite time). The difficulty is that we have not shown that there <u>must</u> be one (and only one) zero for $x(t)$ and $y(t)$ in finite time if $x_0/y_0 \neq \sqrt{a/b}$. To prove the latter proposition, however, one uses an approach that is essentially equivalent to proving Proposition 2.2.1 by apprach (1) of Table 2.IV. Thus, we reach the conclusion that although the state-equation approach to developing force-annihilation-prediction conditions yields the simplest way of guessing the desired conditions, this approach is totally unsatisfactory for proving that the condition is indeed sufficient to guarantee the occurrence of force annihilation in finite time (even for the simple constant-coefficient model (2.2.1)).

Approach (5) consists of showing that one force level may be expressed

as a strictly increasing function of the other one. This monotonicity condition is usually developed, however, by using the state equation. The desired force-annihilation condition may then be readily deduced from such a relationship, but we will defer further discussion of this approach, which is apparently due to HELMBOLD [37], until the next chapter (see Section 3.3).

Let us conclude this section by showing that for the combat model (2.2.1) there must be exactly one zero for $x(t)$ and $y(t)$ in finite time if $x_0/y_0 \neq \sqrt{a/b}$. As in the proof of Proposition 2.2.2, let us multiply the first equation of (2.2.1) by y, the second by x, and add to obtain

$$\frac{d}{dt}(xy) = -(ay^2 + bx^2) \quad . \tag{2.3.4}$$

Similarly from (2.2.1) we also find that

$$\frac{d}{dt}(ay^2 + bx^2) = -4abxy \quad . \tag{2.3.5}$$

Thus, the system of differential equations (2.2.1) is equivalent to

$$\begin{cases} \dfrac{d\pi}{dt} = -\sigma \quad , \\ \\ \dfrac{d\sigma}{dt} = -4ab\pi \quad , \end{cases} \tag{2.3.6}$$

where $\pi = xy$ and $\sigma = ay^2 + bx^2$. It follows that the product of the force levels π satisfies the following differential equation

$$\frac{d^2\pi}{dt^2} - 4\,ab\pi = 0 \quad , \tag{2.3.7}$$

with initial conditions

$$\pi(0) = x_0y_0, \qquad \text{and } \frac{d\pi}{dt}(0) = -(ay_0^2 + bx_0^2) \quad .$$

Solving (2.3.7) we find that

$$\pi(t) = \frac{1}{4\sqrt{ab}} \left\{ - (x_0 \sqrt{b} - y_0 \sqrt{a})^2 e^{2\sqrt{ab}\ t} + (x_0 \sqrt{b} + y_0 \sqrt{a})^2 e^{-2\sqrt{ab}\ t} \right\} \quad (2.3.8)$$

whence it is obvious that $\pi(0) > 0$ but $\pi(t)$ must become negative as $t \to +\infty$ if $x_0/y_0 \neq \sqrt{a/b}$. Thus, we have proved the assertion that there is exactly one finite zero for $x(t)$ and $y(t)$. Let us note, however, that solving (2.3.7) in terms of exponential functions is essentially equivalent to developing (2.2.8), whence our comment that showing that $x_0/y_0 < \sqrt{a/b}$ is sufficient to guarantee force annihilation in finite time by using the state-equation approach (i.e. approach (4) of Table 2.IV) is equivalent to proving Proposition 2.3.1 by approach (1) of Table 2.IV.

Let us finally note that from (2.3.6) we may similarly deduce that

$$\frac{d}{dt}(\sigma^2 - 4ab\pi^2) = 0 \quad , \qquad (2.3.8)$$

which is equivalent to the state equation (2.2.5).

2.4. Constant-Coefficient LANCHESTER-Type Equations for Area Fire.

LANCHESTER [55] also hypothesized that under "conditions of long-range fire with fire concentrated on a certain area," combat between two homogeneous forces could be modelled by

$$\begin{cases} \dfrac{dx}{dt} = -axy & \text{with } x(0) = x_0 , \\ \dfrac{dy}{dt} = -bxy & \text{with } y(0) = y_0 , \end{cases} \tag{2.4.1}$$

where a and b are again called LANCHESTER attrition-rate coefficients. This time, however, such an attrition-rate coefficient represents both the effectiveness of a side's fire and also the vulnerability of enemy targets to that fire. Thus, the a's and b's (i.e. the LANCHESTER attrition rate coefficients) are different in equations (2.2.1) and (2.4.1) and may be related to different physical quantities (<u>see</u> Chapter 5). For simplicity, however, we have chosen to denote, for example, "X's attrition-rate coefficient" as b in both (2.2.1) and (2.4.1), and we caution the reader that b therefore has a different meaning in these two equations.

In this simple combat model (2.4.1), the attrition rate for each force, e.g. $(-dx/dt)$ for the X force, is assumed to be proportional to the product of the numbers of firers and targets. For convenience, let us refer to the equations (2.4.1) as <u>LANCHESTER's equations for area fire</u>,[20] although they have been hypothesized to also apply under other circumstances. In fact, two sets of physical circumstances under which these equations have been hypothesized to apply are:

(C1) both sides use "area" fire and a constant area defense [15,99].

(C2) both sides use "aimed" fire, and target acquisition times are:
(a) inversely proportional to the number of enemy targets, and
(b) the dominant factor in the attrition process [15].

A more complete discussion of these hypotheses is again to be found in the papers by BRACKNEY [15] and WEISS [99] and in Section 2.11 below.

Let us now consider what we can learn from our model (2.4.1) about the dynamics of combat between two homogeneous forces. We will do this again by considering the seven questions (Q1)-(Q7) posed in Section 2.2 above. We begin by again developing (1) the state equation, and (2) the X force level as a function of time, $x(t)$.

To develop the state equation for the combat model (2.4.1), we divide the first equation by the second to obtain the instantaneous (or differential) casualty-exchange ratio

$$\frac{dx}{dy} = \frac{a}{b} . \tag{2.4.2}$$

Separating variables and integrating, we obtain the <u>state equation for LANCHESTER's equations for area fire</u>

$$b\{x_0 - x(t)\} = a\{y_0 - y(t)\} . \tag{2.4.3}$$

We will also refer to (2.4.3) as <u>LANCHESTER's linear law</u>. Solving for y and substituting into the first differential equation of (2.4.1), we obtain the following RICCATI equation for the X force level

$$\frac{dx}{dt} = -bx^2 + \delta_0 x , \tag{2.4.4}$$

where $\delta_0 = bx_0 - ay_0$. For $\delta_0 \neq 0$, a partial fraction expansion yields (see INCE [41, pp. 311-312])

$$-\frac{dx}{x} + \frac{b\,dx}{(bx-\delta_0)} = -\delta_0\,dt\ , \tag{2.4.5}$$

which readily yields our desired result for $x(t)$. For $\delta_0 = 0$, (2.4.4) becomes

$$-\frac{dx}{x^2} = b\,dt\ , \tag{2.4.6}$$

which is also readily integrated. Hence, we find that

$$x(t) = \begin{cases} x_0\left[\dfrac{bx_0 - ay_0}{bx_0 - ay_0\exp[-(bx_0-ay_0)t]}\right] & \text{for } bx_0 \neq ay_0, \\[2ex] \dfrac{x_0}{1 + bx_0 t} & \text{for } bx_0 = ay_0. \end{cases} \tag{2.4.7}$$

Later, it will be of interest to consider the variable coefficient version of (2.4.1) for which no state equation such as (2.4.3) generally holds. With this in mind, we would like to be able to develop (2.4.7) by a method that does not involve the state equation (2.4.3) and can consequently be extended to the variable-coefficient case. We have discussed such a point previously in Section 2.2 above. Accordingly, we again differentiate the first equation of (2.4.1) with respect to t and combine the result with the second equation to obtain a second order nonlinear ordinary differential equation that contains only the X force level, namely

$$\frac{d^2x}{dt^2} - \frac{1}{x}\left(\frac{dx}{dt}\right)^2 + bx\,\frac{dx}{dt} = 0\ , \tag{2.4.8}$$

with initial conditions

$$x(0) = x_0 \,, \qquad \text{and} \qquad \frac{dx}{dt}(0) = -ax_0y_0 \,.$$

We will call (2.4.8) the X <u>force-level equation</u>. It is the analogue of equation (2.2.10). This nonlinear differential equation (2.4.8) is one of fifty standard forms for a certain class of nonlinear second order equations.[23] Unfortunately, there apparently is no analytical technique for solving (2.4.8) directly, and thus hope for the analytical treatment of the variable coefficient version of (2.4.1) appears dim. However, the term $1/x$ is an integrating factor for (2.4.8), and we find that

$$\frac{d}{dt}\left(\frac{1}{x}\frac{dx}{dt}\right) + b\,\frac{dx}{dt} = 0 \,, \tag{2.4.9}$$

whence integration yields the RICCATI equation (2.4.4). Thus, without use of any approximation, the X force-level equation (2.4.8) is not as useful as the corresponding equation (2.2.10) was for the model (2.2.1).

The decay of the X and Y force levels is plotted in Figures 2.8 and 2.9. For convenience, we record these results here as

$$x(t) = \begin{cases} x_0\left[\dfrac{bx_0 - ay_0}{bx_0 - ay_0\,\exp[-(bx_0-ay_0)t]}\right] & \text{for } bx_0 \neq ay_0, \\[2ex] \dfrac{x_0}{1 + bx_0t} & \text{for } bx_0 = ay_0, \end{cases} \tag{2.4.10}$$

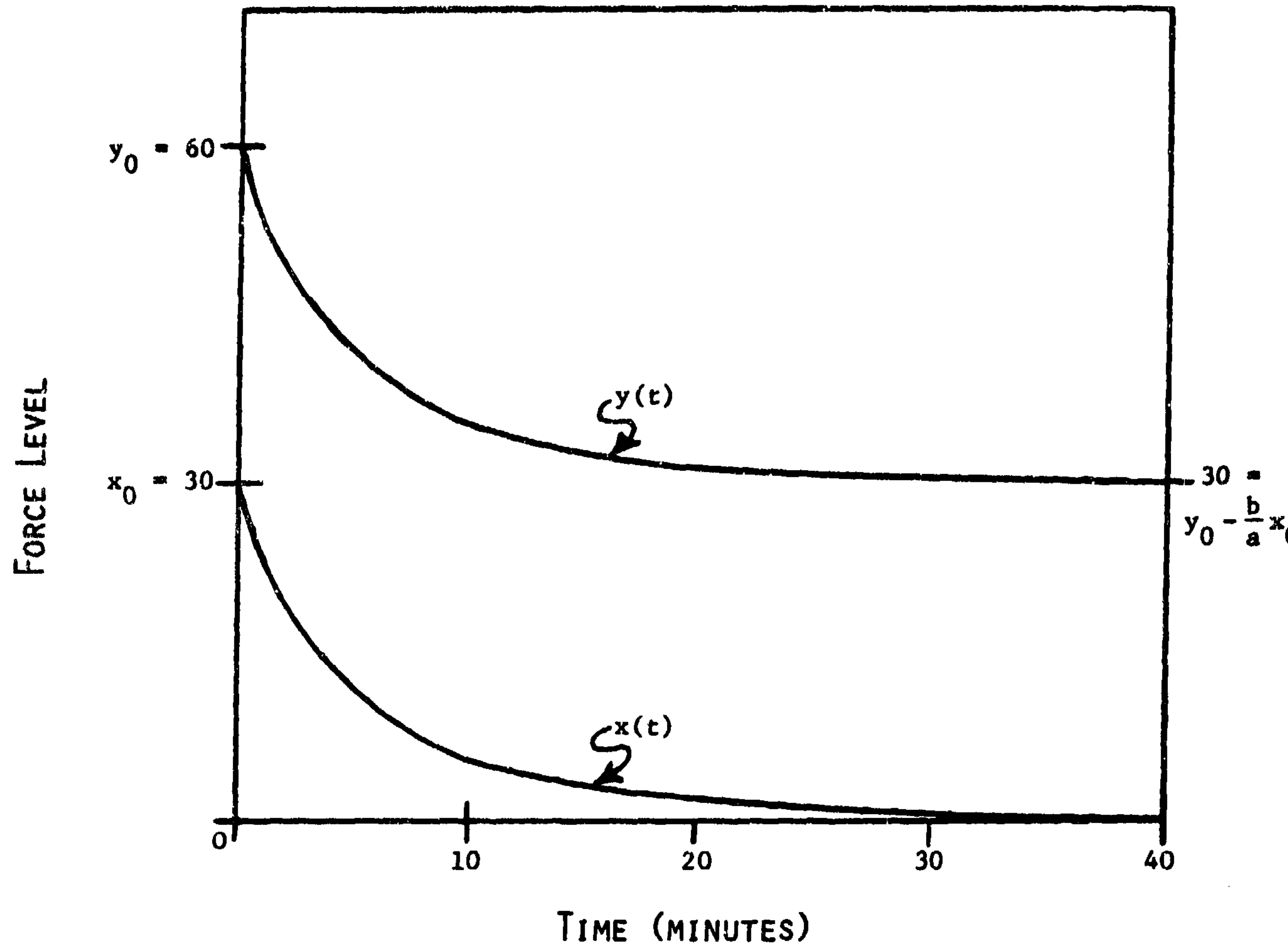


Figure 2.8. Force-level trajectories of X and Y forces for combat modelled by LANCHESTER's equations for area fire. For these calculations, a = 0.004 X casualties/(minute • number of X combatants • number of Y combatants) and b = 0.004 Y casualties/(minute • number of X combatants • number of Y combatants).

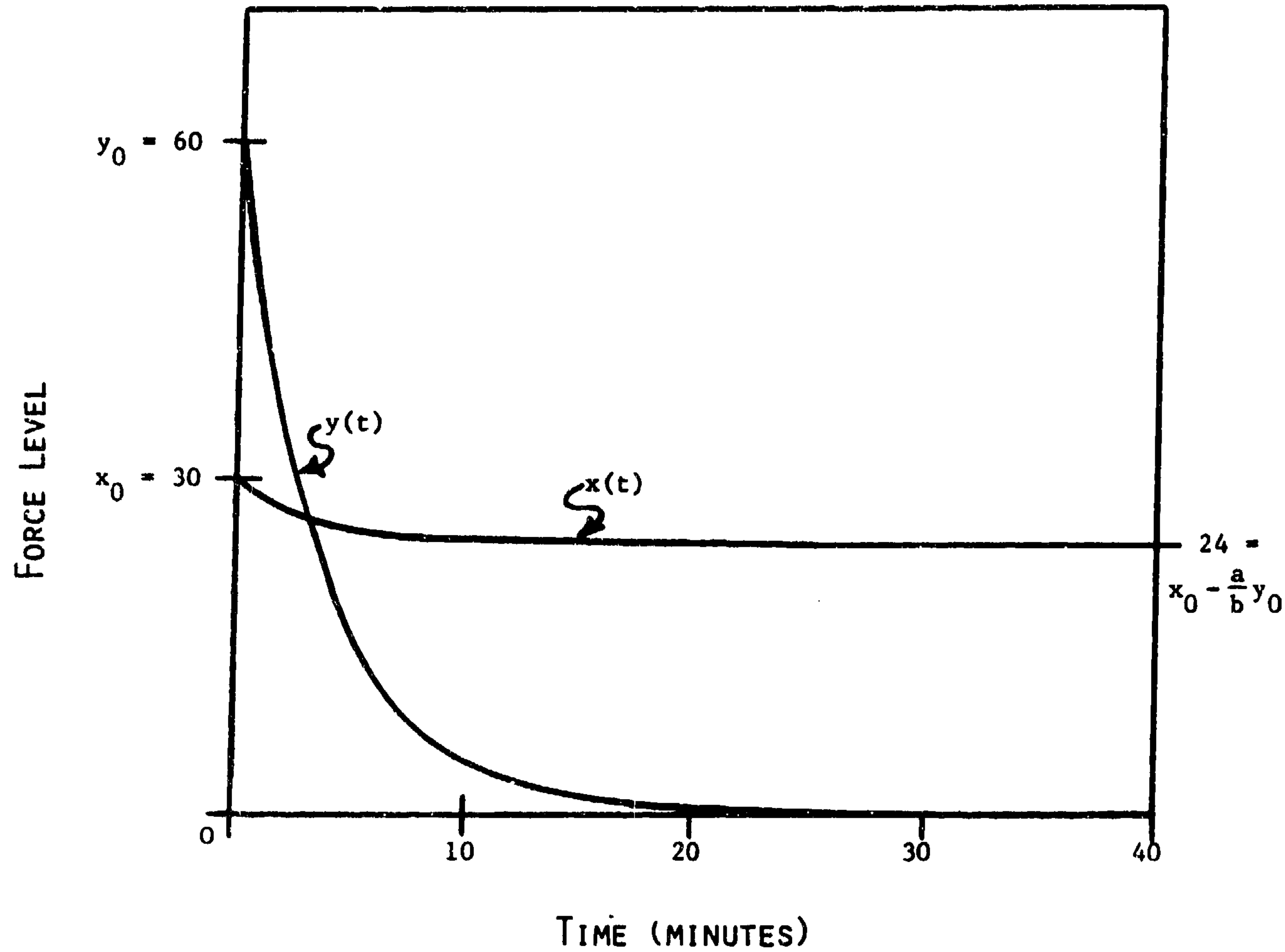


Figure 2.9. Force-level trajectories of X and Y forces for combat modelled by LANCHESTER's equations for area fire. For these calculations, $a = 0.001$ X casualties/(minute · number of X combatants · number of Y combatants) and $b = 0.01$ Y casualties/(minute · number of X combatants · number of Y combatants).

and

$$y(t) = \begin{cases} y_0 \exp[-(bx_0-ay_0)t]\left[\dfrac{bx_0 - ay_0}{bx_0 - ay_0 \exp[-(bx_0-ay_0)t]}\right] & \text{for } bx_0 \neq ay_0 , \\ \dfrac{y_0}{1 + ay_0 t} & \text{for } bx_0 = ay_0 . \end{cases} \tag{2.4.11}$$

In Figure 2.8 the smaller force is seen to be annihilated. In contrast to the model (2.2.1), however, force annihilation is seen to be an asymptotic result, i.e. it takes "infinite time" to occur. Thus, $x(t)$ and $y(t) > 0$ for all finite t, and we do not have to "turn off" the equations (2.4.1) to avoid negative force levels as we had to do for the model (2.2.1) [see in this respect (2.2.2)]. In Figure 2.8 the smaller force is annihilated, while in Figure 2.9 the larger one is.

To more clearly exhibit the parametric dependence of the force-level trajectories, we again "normalize," for example, the X force level by considering the fractional X force level, namely $x(t)/x_0$, given by

$$\frac{x(t)}{x_0} = \frac{\rho - 1}{\rho - e(t)} , \tag{2.4.12}$$

where $\rho = bx_0/ay_0$ and $e(t) = \exp[-ay_0 t(\rho-1)]$. From (2.4.12) we see that the X force level depends on the following three quantities (although the model (2.4.1) contains the four independent parameters a, b, x_0, and y_0):

(1) initial force ratio, $u_0 = x_0/y_0$,

(2) relative fire effectiveness, $R = a/b$,

(3) initial volume of enemy fire, $V_0 = ay_0$.

The initial force ratio u_0 and the relative fire effectiveness[24] R are the same two relative quantities that we encountered in our study of the model (2.2.1), whereas the initial volume of enemy fire V_0 is an absolute quantity that corresponds to the intensity of combat $I = \sqrt{ab}$ for the model (2.2.1).

Let us now consider the determination of who will "win" the battle. Again, for simplicity, we will consider here only a "fight-to-the-finish," with a more realistic model of battle termination being considered in Section 2.8 below. From considering (2.4.10) and (2.4.11), we can make a number of important observations: (1) $x(t)$ and $y(t) > 0$ for all finite $t \geq 0$, (2) $\lim_{t \to +\infty} x(t) = 0$ if and only if $x_0/y_0 < a/b$, and $\lim_{t \to +\infty} x(t) = 0$ if and only if $\lim_{t \to +\infty} y(t) = y_0 - (b/a)x_0$. Thus, we have shown

> PROPOSITION 2.4.1: Y will win a fight-to-the-finish if and only if $x_0/y_0 < a/b$. The time required to annihilate X is not finite, however.

Furthermore,

> PROPOSITION 2.4.2: For the model (2.4.1), we have $x(t)$ and $y(t) > 0$ for all finite $t \geq 0$. Consequently, both $x(t)$ and $y(t)$ are always strictly decreasing, positive functions.

As we have pointed out in section 2.2 above (see also Section 1.3), most aggregated models of ground combat (for example, ATLAS) use the force

ratio to determine both casualty rates and also FEBA movement. Consequently, it is of considerable interest to investigate how the force ratio, e.g. $u = x/y$, changes during the course of battle for our simple combat model (2.4.1). We first observe that in general logarithmic differentiation of the force ratio, $u = x/y$, yields

$$\frac{1}{u}\frac{du}{dt} = \frac{1}{x}\frac{dx}{dt} - \frac{1}{y}\frac{dy}{dt}, \tag{2.4.13}$$

whence for the model (2.4.1) we obtain[25]

$$\frac{du}{dt} = bx(u - \frac{a}{b}) . \tag{2.4.14}$$

Thus, we see that unlike the case of the model (2.2.1), there is no first order differential equation involving just the force ratio for the model (2.4.1). We can artificially achieve this situation, however, by letting $\tau = b \int_0^t x(s)\, ds$, and then

$$\frac{du}{d\tau} = u - \frac{a}{b} . \tag{2.4.15}$$

Following an analysis similar to that given in Section 2.2 for the force-ratio equation (2.2.16), we can easily prove Proposition 2.4.3.

PROPOSITION 2.4.3: If $du/dt(0) < 0$, then $du/dt(t) < 0$ for all $t \geq 0$.

Thus, if $u_0 = x_0/y_0 < a/b$, the force ratio will always decrease during the

course of battle; it will remain constant if and only if $x_0/y_0 = a/b$ although the force levels continuously decay, of course [e.g. $x(t) = x_0/(1 + bx_0t)$]. It is very important to note that $du/dt < 0$ for all $t \geq 0$ does not in this case imply that $u(t) \to 0$ in finite time, since it is no longer true that $du/dt(t) \leq du/dt(0)$ when $du/dt(0) < 0$.

From (2.4.10) and (2.4.11) it is clear that neither side can ever be annihilated in finite time. Thus, our model says that a fight-to-the-finish will be of infinite duration. We do find from (2.4.10) that for $bx_0 \neq ay_0$ it takes time t_f for the X force level to decay to a given value x_f, namely

$$t_f = \frac{1}{ay_0(1-\rho)} \ln(\rho + \frac{x_0}{x_f}[1-\rho]) , \tag{2.4.16}$$

where $\rho = bx_0/ay_0 \neq 1$ and the following restrictions must be placed on x_f:

$$\begin{cases} 0 \leq x_f \leq x_0 & \text{for} \quad \rho < 1 , \\ x_0 - \frac{a}{b} y_0 \leq x_f \leq x_0 & \text{for} \quad \rho > 1 . \end{cases}$$

The number of survivors, expressed as a fraction of initial strength, for the winner (here Y for $x_0/y_0 < a/b$) of such a fight-to-the-finish is readily obtained from the state equation (2.4.3) to be

$$\frac{y_f}{y_0} = 1 - \frac{b}{a}\frac{x_0}{y_0} , \tag{2.4.17}$$

where y_f denotes the final Y force level at $t = +\infty$. This equation shows us quite clearly that fractional casualties are determined entirely by relative factors. For any other (nonnegative) value of the X force level, we (of course) have

$$\frac{y}{y_0} = 1 - \frac{b}{a}\left(\frac{x_0}{y_0} - \frac{x}{y_0}\right) . \tag{2.4.18}$$

The principal results that we have developed above are summarized in Table 2.V.

TABLE 2.V. Summary of Principal Results for LANCHESTER's Model of Combat with Area Fire by Both Sides.

LANCHESTER's Equations for Area Fire

$$\begin{cases} \dfrac{dx}{dt} = -axy & \text{with } x(0) = x_0 \\ \dfrac{dy}{dt} = -bxy & \text{with } y(0) = y_0 \end{cases}$$

Differential Casualty-Exchange Ratio, $\frac{dx}{dy}$: $\frac{dx}{dy} = \frac{a}{b}$

State Equation: $a\{y_0 - y(t)\} = b\{x_0 - x(t)\}$

Differential Equation Satisfied by the X Force Level:

$$\frac{d^2x}{dt^2} - \frac{1}{x}\left(\frac{dx}{dt}\right)^2 + bx\frac{dx}{dt} = 0 .$$

with initial conditions

$$x(0) = x_0 \qquad \text{and} \qquad \frac{dx}{dt}(0) = -ax_0y_0$$

X Force Level:

$$x(t) = \begin{cases} x_0\left[\dfrac{bx_0 - ay_0}{bx_0 - ay_0\exp[-(bx_0-ay_0)t]}\right] & \text{for } bx_0 \neq ay_0 \\ \dfrac{x_0}{1 + bx_0t} & \text{for } bx_0 = ay_0 \end{cases}$$

Differential Equation Satisfied by the Force Ratio, $u = \frac{x}{y}$:

$$\frac{du}{dt} = bx\left(u - \frac{a}{b}\right) \qquad \text{with } u(0) = \frac{x_0}{y_0}$$

Force-Annihilation-Prediction Condition: X will be annihilated (in infinite time) if and only if $x_0/y_0 < a/b$.

*2.5. A Further Look at the Area-Fire Model.

In this section we present a more in depth analysis of LANCHESTER's model for area fire (2.4.1). In particular, we will consider the following topics:

(T1) a solution approach that can be generalized to cases of time-dependent attrition-rate coefficients,

and (T2) determining the qualitative behavior of the force-level trajectories for the model (2.4.1) without having to explicitly solve the system of differential equations.

As note above, for the general case of time-dependent attrition-rate coefficients, there is no state equation of the form $S(x,y) = 0$. With this fact in mind, let us seek a method of solving (2.4.1) that does not depend on using such a state equation. Accordingly, we will develop a method of solving (2.4.1) that has this property and consequently for cases of time-dependent attrition-rate coefficients, will allow us to determine two approximate solutions that many times bound the exact solution, for example, for the X force level.

We begin by providing motivation for a key transformation that "linearizes" our nonlinear combat model. Let us rewrite the RICCATI equation satisfied by the X force level $x(t)$, namely

$$\frac{dx}{dt} = - bx^2 + \delta_0 x \ , \tag{2.5.1}$$

where $\delta_0 = bx_0 - ay_0$. Since there is no constant term on the right-hand side of (2.5.1), it is a special case of a particular kind of RICCATI equation called a BERNOULLI equation (see, for example, HILLE [39 , pp. 104-105]). The nonlinear BERNOULLI equation, moreover, can be transformed

to a linear equation by a substitution for the dependent variable. For (2.5.1) this substitution takes the form $w = 1/x$. Let us therefore make the substitution

$$w = 1/x \qquad \text{and} \qquad z = 1/y \tag{2.5.2}$$

in (2.4.1) to obtain

$$\begin{cases} \dfrac{dw}{dt} = \dfrac{aw}{z} & \text{with } w(0) = 1/x_0 , \\ \\ \dfrac{dz}{dt} = \dfrac{bz}{w} & \text{with } z(0) = 1/y_0 . \end{cases} \tag{2.5.3}$$

The first equation of (2.5.3) may be rearranged and differentiated to yield

$$\frac{d}{dt}\left\{\frac{1}{aw}\frac{dw}{dt}\right\} = -\frac{1}{z^2}\frac{dz}{dt} . \tag{2.5.4}$$

We may also manipulate (2.5.3) to obtain that $-(1/z^2)\, dz/dt = (b/a)d(1/w)dt$, whence (2.5.4) becomes

$$\frac{d}{dt}\left\{\frac{1}{aw}\frac{dw}{dt} - \frac{b}{aw}\right\} = 0 . \tag{2.5.5}$$

Integrating (2.5.5), we obtain

$$\frac{dw}{dt} + (bx_0 - ay_0)w = b , \tag{2.5.6}$$

whence a second integration yields

$$w(t) = \begin{cases} \dfrac{bx_0 - ay_0 \exp[-(bx_0 - ay_0)t]}{x_0(bx_0 - ay_0)} & \text{for } bx_0 \neq ay_0 , \\ \\ \dfrac{1 + bx_0 t}{x_0} & \text{for } bx_0 = ay_0 . \end{cases} \tag{2.5.7}$$

Recalling (2.5.2), we readily obtain (2.4.7) from (2.5.7). Moreover, this solution approach may be used to develop some very useful approximations in cases of time-dependent attrition-rate coefficients, since we did not make essential use of the state equation (2.4.3).

Let us next determine some important solution properties for the model (2.4.1) without having to develop an explicit solution. We begin by examining the qualitative behavior of the X force level $x(t)$ as determined directly from the RICCATI equation (2.5.1). We will show that much valuable information (e.g. force-annihilation prediction) about the force-level trajectories of the model (2.4.1) may be obtained directly from (2.5.1) without explicitly solving for $x(t)$. Let us accordingly focus on the RICCATI equation (2.5.1)

$$\frac{dx}{dt} = -bx^2 + (bx_0 - ay_0)x \; . \tag{2.5.1}$$

It seems appropriate to call dx/dt the <u>force-level velocity</u>. Let us denote the two roots of the equation $bx^2 - (bx_0 - ay_0)x = 0$ as x_1 and x_2, with $x_1 = x_0 - (a/b)y_0$ and $x_2 = 0$. Then the maximum of dx/dt considered as a function of x occurs at $\bar{x} = (x_1 + x_2)/2$. The corresponding RICCATI equation satisfied by the Y force level $y(t)$ is

$$\frac{dy}{dt} = -\, ay^2 + (ay_0 - bx_0)y \; , \tag{2.5.8}$$

and we similarly define y_1 and y_2 with $y_1 = y_0 - (b/a)x_0$ and $y_2 = 0$. We observe that $x_1 = -(a/b)y_1$ so that x_1 and y_1 always have opposite signs except when they are both equal to zero. There are now three cases to be considered: (I) $x_0/y_0 < a/b$, (II) $x_0/y_0 = a/b$, and (III) $x_0/y_0 > a/b$.

In Figure 2.10 the force-level velocity is plotted against the force level for each of the X and Y forces in Case (I): $x_0/y_0 < a/b$. The "direction" of movement for the force level is shown in Figure 2.10 by means of arrows drawn along the force-level axis, with the length of the arrow reflecting the magnitude of the force-level velocity. In this case, $x_1 = x_0 - (a/b)y_0 < x_2 = y_2 < y_1 = y_0 - (b/a)x_0$. We always have $|x_1| < x_0$ and $|y_1| < y_0$. From Figure 2.10 it is clear that $y(t) \to y_0 - (b/a)x_0$ and $x(t) \to 0$ as $t \to +\infty$, and also that $x(t)$ and $y(t) \geq 0$ for all $t \geq 0$. Thus, by plotting the force-level velocity versus the force level for each of the combatants, the qualitative behavior of the force levels becomes obvious. In Case (II) both $x(t)$ and $y(t) \to 0$ as $t \to +\infty$. Case (III) is symmetric to Case (I), with the roles of X and Y interchanged. Thus, we see that in all cases $x(t)$ and $y(t) \geq 0$ for all $t \geq 0$.

Let us now show that for the model (2.4.1) [without any modification of the right-hand sides, cf. (2.2.2)] $x(t)$ and $y(t) > 0$ for all finite $t \geq 0$. The easiest way to do this without explicitly solving the differential equations is to introduce functions $\pi(t)$ and $\sigma(t)$, analogous to those introduced in Section 2.3 above. To this end, let us multiply the first equation of (2.4.1) by y, the second by x, and add to obtain

$$\frac{d}{dt}(xy) = -xy(ay + bx) . \qquad (2.5.9)$$

Similarly,

$$\frac{d}{dt}(ay + bx) = -2abxy . \qquad (2.5.10)$$

Let us rewrite the above as

$$\frac{d\pi}{dt} = -\pi\sigma , \qquad \text{and} \qquad \frac{d\sigma}{dt} = -2ab\,\pi , \qquad (2.5.11)$$

where $\pi = xy$ and $\sigma = ay + bx$. We observe that as a consequence of (2.5.11) we have

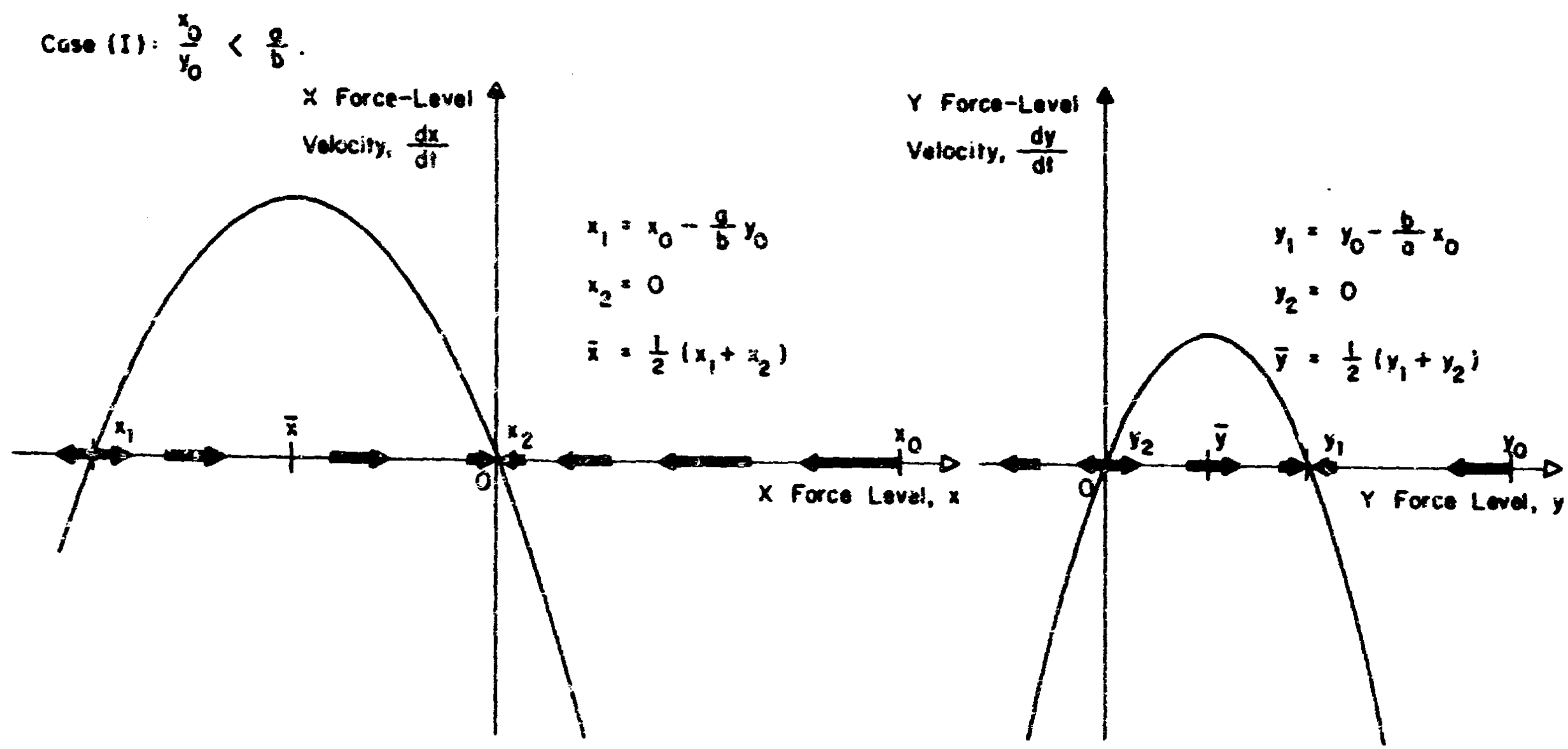


Figure 2.10. Force-level velocity as a function of the force level for the X and Y forces, Case (I): $x_0/y_0 < a/b$. For these calculations $x_0 = 100$, $y_0 = 100$, $a = 0.002$ X casualties/(minute · number of X combatants · number of Y combatants), and $b = 0.001$ Y casualties/(minute · number of X combatants · number of Y combatants).

$$\frac{d}{dt}(\sigma^2 - 4ab\pi) = 0 , \qquad (2.5.12)$$

which is equivalent to the state equation (2.4.3).

By considering the first equation of (2.5.11), we will now show that $x(t)$ and $y(t) > 0$ for all finite $t \geq 0$. Recall that we have shown above by considering the two RICCATI equations (2.5.1) and (2.5.8) that $x(t)$ and $y(t) \geq 0$ for all $t \geq 0$. It follows that $\pi(t) \geq 0$ and from (2.5.11) that $\sigma(t)$ is a decreasing function of time. Hence, $\sigma(t) \leq \sigma_0 = \sigma(0)$ for all $t \geq 0$ so that

$$\frac{d\pi}{dt} \geq -\pi\sigma_0 ,$$

whence

$$\pi(t) \geq \pi_0 \, e^{-\sigma_0 t} . \qquad (2.5.13)$$

This last result (2.5.13) shows that $x(t)$ and $y(t) > 0$ for all finite t, since $\pi(t) > 0$ for all finite $t \geq 0$. Thus, we have proven Proposition 2.4.2 without explicitly solving the equations (2.4.1). It is desirable, however, for extending this result to the case of time-dependent attrition-rate coefficients to use the following argument.

Another (however, much more important) way to prove Proposition 2.4.2 is to consider the system of equations (2.5.3) satisfied by $w = 1/x$ and $z = 1/y$. We will prove the following proposition (which is equivalent to Proposition 2.4.2).

PROPOSITION 2.5.1: The solution $w(t)$, $z(t)$ to (2.5.3) is positive and bounded for all finite $t \geq 0$.

PROOF: First, we show that no component of the solution to (2.5.3) can become negative by passing through zero. We prove this by contradiction: let $t_1 = \inf\{t \mid w(t) = 0 \text{ or } z(t) = 0 \text{ for } t > 0\}$ and assume that t_1 is finite. Then $w(t)$ and $z(t) > 0$ for $t \in [0,t_1)$, and $dw/dt(t)$ and $dz/dt(t) > 0$ by (2.5.3), which is impossible if a component of the solution is to have a finite zero. The only other way in which a component of the solution can become negative would be for it first to become infinite. Let $t_2 = \inf\{t \mid w(t) = \infty \text{ or } z(t) = \infty \text{ for } t > 0\}$. We will now show that it is impossible for t_2 to be finite. If this were indeed the case, then $w(t)$ and $z(t) > 0$ for $t \in [0,t_2)$ so that $dw/dt(t)$ and $dz/dt(t) > 0$. It suffices to show that $w(t_2)$ cannot be unbounded for any finite $t_2 > 0$. Let us note that $z(t) \geq z_0 > 0$ for $t \geq 0$. Hence, we have from (2.5.3) that

$$\frac{dw}{dt} \leq \frac{a}{z_0} w \ .$$

Thus, for any finite $t > 0$, we have

$$w(t) \leq w_0 \exp(\frac{a}{z_0} t) < \infty \quad . \tag{2.5.14}$$

Hence, the solution to (2.5.3) is positive and bounded for all finite $t \geq 0$. <u>Q.E.D.</u>

2.6. Shortcomings of Lanchester's Original Models.

Viewed in the light that LANCHESTER [55] developed his very simple models of combat (2.1.1), (2.1.3), and (2.1.8) to provide insight into the dynamics of combat under "modern conditions" and to quantitatively justify the principle of concentration, LANCHESTER's simple differential equation models are quite reasonable. They yield results that are in consonance with military judgement. Although such simple analytical models can provide valuable insights into the dynamics of combat, they are far too simple to be able to solve by themselves any specific operational problem. Thus, from the point of view of a weapon-system designer or defense planner, who is interested in more than just insights[26], different demands are made on a model. In particular, the "realities of the real world" must be "adequately" treated in the model in order that sound recommendations be based on the information that it generates. Accordingly, we will now examine what factors are not "adequately" treated in LANCHESTER's original models, i.e. their shortcomings.

Speaking about the shortcomings of LANCHESTER's classic combat formulations, WEISS [98, p. 15] has eloquently stated,

> "While we should, perhaps, be more pleased that such simple formulae yield reasonable results than critical because of the elements omitted from them, we must look beyond the LANCHESTER expressions to see how they differ from reality, and what may be added to them"

With this in mind, we have listed some of the major shortcomings of LANCHESTER's original models (2.1.1), (2.1.3), and (2.1.8) in Table 2.VI. These shortcomings are listed roughly in order of decreasing importance, with the most important ones appearing first in the list.

TABLE 2.VI. Shortcomings of LANCHESTER's Original Models

SHORTCOMINGS:

1. Constant attrition-rate coefficients
2. No force movement (e.g. no advance or retreat of forces)
3. Homogeneous forces
4. Battle termination not modelled
5. No element of chauce
6. Not verified by history
7. No way to predict attrition-rate coefficients
8. Tactical decision processes not considered
9. Battlefield intelligence not considered
10. Command, control, and communications not considered
11. Logistics aspects not considered
12. Suppressive effects of weapons not considered
13. Effects of terrain not considered
14. Spatial variations in force capabilities not considered
15. No replacements or withdrawals
16. Symmetric form of attrition
17. Target priority/fire allocation not explictly considered
18. Target acquisition force-level independent in modern-warfare model
19. All troops assumed to fire in combat
20. Noncombat losses (e.g. surrenders, desertions) not considered

Let us now briefly discuss the first ten shortcomings of LANCHESTER's classic models given in Table 2.VI.

(S1) Constant attrition-rate coefficients essentially mean that the kill rate of each and every weapon system doesn't change over time due to changes in range between target and firer, target posture, firing rate, vulnerability of the target, target acquisition rate, etc.

(S2) No provision is explicitly made for movement, retreat or advance. In particular, the movement of contact zones (i.e. FEBA movement) is not considered.

(S3) All forces on one side are considered to be the same. In combined arms engagements, one usually has various different force types, such as infantry, artillery, armor, mortars, mechanized infantry combat vehicles, tactical aircraft, etc. Also, there are other factors such as minefields, fortifications, barriers, smoke, etc. Furthermore, spatial variations in the effectiveness of forces are not considered.

(S4) No rules for battle termination are given. WEISS [98, p. 16] emphasized that "engagements that continue until one side is wiped out are rare. Retreat begins when the number of casualties approaches the order of 10%."

(S5) The equations are deterministic and do not portray the random nature of combat. Many of the factors in combat are of a random nature, and the uncertainty[27] in battle outcome is lost when one models combat with such deterministic equations.

(S6) A priori we have no confidence that combat (even in a gross sense) actually behaves as postulated by LANCHESTER. Empirical verification would greatly enhance the acceptability of such a basis for operational models by users and decision makers.

(S7) One doesn't know how to develop numerical values for the attrition-rate coefficients such that the performance characteristics of the weapon systems and the operating environment are adequately reflected in the model.

(S8) Decisions to initiate combat, commit forces and/or reserves, allocate fires, allocation of effort searching for targets, etc. are not explictly considered.

(S9) The ability to locate and identify targets, correctly sense killed targets, etc. are not explictly considered.

(S10) The passing of information up and down the chain of command is not considered.

We could go on and on. However, Table 2.VI and our brief discussion here should give the reader some flavor of the shortcomings of LANCHESTER's classic models.

The reader should recognize that many such shortcomings are not strictly limited to only Lanchester-type models. If one doesn't know how, for example, command and control influences weapon-system kill rates in a particular combat environment, then this is not necessarily a shortcoming of Lanchester-type models. It will also apply to the firepower-score and Monte-Carlo-simulation combat-modelling approaches. The author believes that if a combat process can be modelled at all, then it can ultimately be modelled with a differential equation model of some type.

In spite of all these shortcomings, the amazing thing is that such simple differential-equation models (or their equivalent) are frequently used even today. It is frequently the case, however, that one does not realize that the combat model he is using either is equivalent to or may be most fruitfully viewed as a differential combat model (see, for example, Chapter 8 below).

From the point of view of the subsequent development and enrichment of differential-equation models of combat (i.e. the so-called Lanchester theory of combat), the above shortcomings of LANCHESTER's original 1914 models have played a central role. Namely, subsequent developments in the Lanchester theory of combat have evolved to overcome these shortcomings.

2.7. Subsequent Development of the LANCHESTER Theory of Combat: A Preview of Things to Come.

As we have just discussed in the previous section, the development of the so-called LANCHESTER theory of combat is probably best understood by considering the shortcomings of LANCHESTER's original 1914 models. Various authors from the 1940's on have subsequently sought to overcome the shortcomings listed in Table 2.VI above, and these individuals have accordingly made various extensions to LANCHESTER's classic combat models. A list of such extensions is given in Table 2.VII. The extensions listed in Table 2.VII are given in more or less chronological order, with the reference(s) given representing in most cases the earliest work on the topic known to this author. References available in the open, unclassified literature are emphasized.

Let us now make some remarks about the various extensions listed in Table 2.VII. The first extensions of LANCHESTER's [55] original work appeared in MORSE and KIMBALL's classic book [64], which reports various investigations undertaken during World War II by American wartime analysis groups. In particular, replacements were added to a model of aggregated force combat, and some implications of the resultant model were studied in [64] (see also KARNS [47]). Equations (both the forward Kolmogorov equations and also "random walk" ones) for a stochastic combat-attrition process were developed, and results from the stochastic model were compared with those from the usual deterministic model in the special case of very few combatants on each side. R. SNOW [78] summarized and extended work done at RAND in the late 1940's. In particular, he examined a LANCHESTER-type, MARKOV-chain model of combat and heterogeneous-force combat formulations. Both the assumptions for LANCHESTER-type combat

TABLE 2.VII. Extensions of LANCHESTER's Classic Combat Models

EXTENSIONS:†

1. Replacements (an/or withdrawals) [47; 64]
2. Heterogeneous forces [78]
3. Inclusion of random effects in the attrition process [64; 78]
4. FEBA movement considered [28; 65; 99]
5. Fire-support effects included [28]
6. Optimization of tactical decisions [28]
7. Comparison with historical data [24; 32; 99]
8. Attrition structures other than LANCHESTER's classic models [15; 36]
9. Unsymmetric formulations for attritions [15; 22]
10. Time- (or range-) dependent attrition-rate coefficients [8; 99]
11. Operational losses considered [3]
12. Rough effects of intelligence and command and control [73]
13. Attrition-rate coefficients that depend on force sizes [36]
14. Models of guerrilla warfare activities [22; 72]
15. Prediction of attrition-rate coefficients [4; 9; 20]
16. Noncombat losses (e.g. surrenders and desertions) [72]
17. Suppressive effects of weapons [72]
18. Modelling of battle termination [37; 102]
19. Interfacing with high-resolution Monte Carlo simulations [20]
20. Large-scale, complex planning models [19; 26]

†Numbers in brackets refer to references at the end of this chapter.

between heterogeneous forces and analytical solution procedures were considered by SNOW [78], although the special structure of the combat equations was not fully exploited for developing analytical solutions in this pioneering work.

The RAND memorandum by GAIMBONI, MENGEL, and DISHINGTON [28] contains a number of pioneering extensions of LANCHESTER's classic combat formulations: (E1) FEBA-movement modelling, (E2) inclusion of fire-support (particularly tactical airpower) effects, and (E3) optimization of the time-sequential allocation of aircraft to tactical targets. This report memorandum is still worthwhile reading today, even though it was written in 1951. MULHOLLAND and SPECHT [65] examined some World War II data and developed a rough model for FEBA movement in theater-level operations (see also WEISS [99]). Pioneering efforts at comparing the theoretical predictions of LANCHESTER-type models with historical data have been by J. ENGEL [24] and H. K. WEISS [99] (see also the work by R. L. HELMBOLD [32-35; 37]).

A benchmark paper, which is still worthwhile reading today although it is somewhat inaccessible, is H. K. WEISS's 1957 paper, "Lanchester-Type Models of Warfare." Many innovative ideas were introduced, including the following: (1) range-dependent attrition-rate coefficients, (2) comparison of model results with historical data, (3) a model of combat among small groups, (4) a model of FEBA movement, and (5) a differential-game examination of optimal fire-support strategies. WEISS's [99] paper is probably the second most referenced paper in the field after LANCHESTER's original paper. Furthermore, all of H. K. WEISS's work has been characterized by imaginative innovation, coupled with deep insights into the scientific analysis of combat operations.

Other models for the mutual attrition of two homogeneous forces in combat have been proposed by BRACKNEY [15] and HELMBOLD [36]. BRACKNEY [15] introduced target acquisition considerations and hypothesized that the time to acquire a target is related to the target's tactical posture. HELMBOLD [36] has proposed a modification of LANCHESTER's equations for modern warfare, which incorporates inefficiencies of scale for the larger force when force sizes are grossly unequal. S. BONDER [8] did the pioneering work on the prediction of attrition-rate coefficients from weapon-system performance characteristics (see also BONDER [9; 10] and BARFOOT [4]), and, motivated by such developments, he examined the effects of range-dependent attrition-rate coefficients and mobility on battle outcome.

Operational losses were considered by BACH, DOLANSKY, and STUBBS [3], who showed that if operational losses were "large enough," then it would no longer be "beneficial" to concentrate forces (i.e. friendly casualties would increase if more friendly forces were initially committed to battle). LANCHESTER-type models of guerrilla-warfare engagements were considered by DEITCHMAN [22] and Schaffer [72]. DEITCHMAN [22] developed a LANCHESTER-type model of an ambush in order to explain the observed high overall force ratios of regulars to guerrillas insurgency operations. WEISS's [99] model for combat among small groups is DEITCHMAN's point of departure. SCHAFFER [72] later developed models of several types of guerrilla-warfare engagements in insurgency warfare. He considered non-combat losses (such as surrenders and desertions) and included suppressive effects for supporting weapons in several of these models.

The above very rough sketch and Table 2.VII should give the reader a general idea of the development of the so-called LANCHESTER theory of combat. Although we haven't discussed every reference cited in Table 2.VII, we have touched upon the high points. Figure 2.11 depicts the

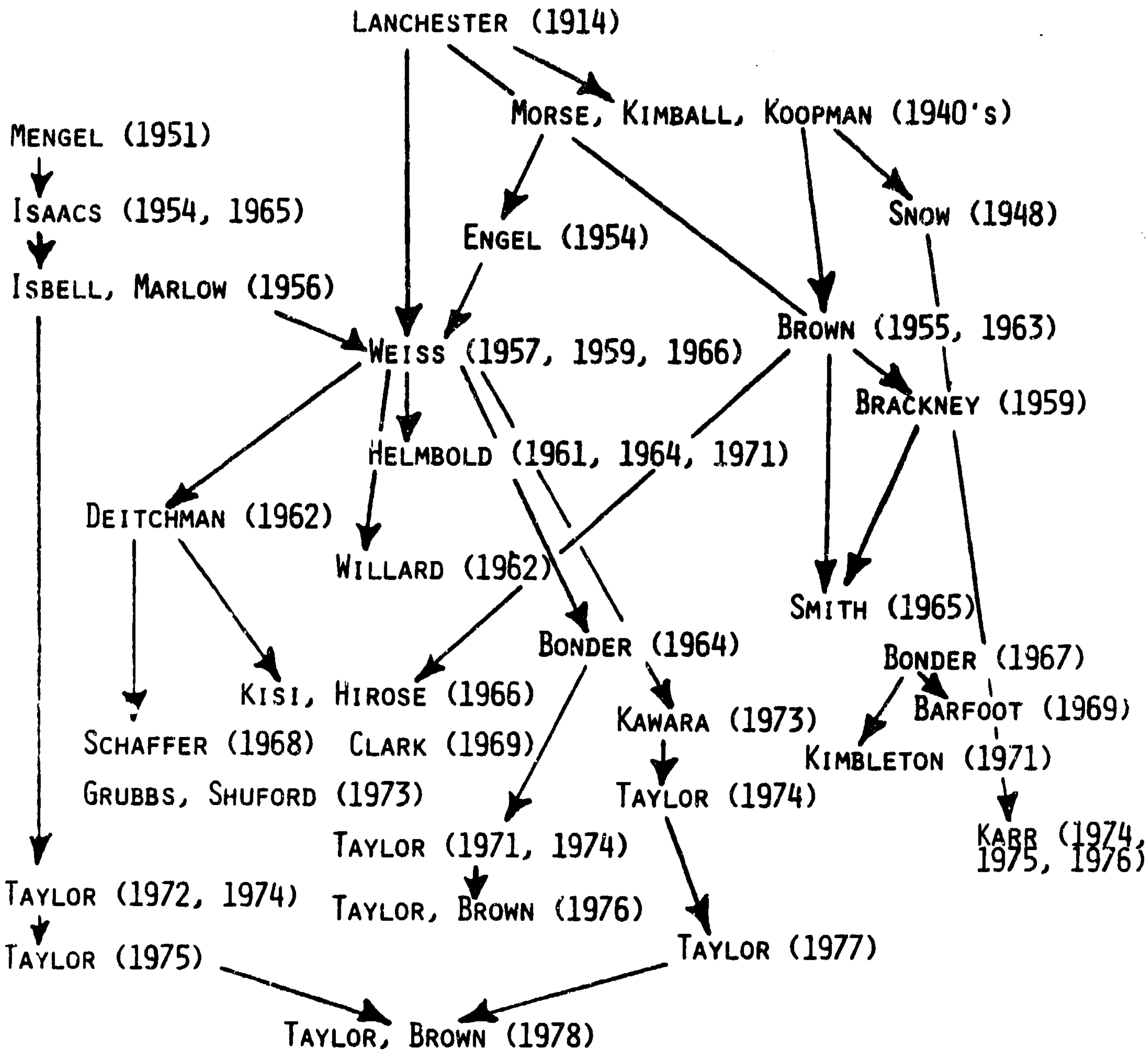


Figure 2.11. Chronology of developments in LANCHESTER theory of combat.

chronology of these developments. In this figure, the arrows depict this author's best guess as to how the works of various authors have influenced each other.

Another way to look at developments in the LANCHESTER theory of combat is to classify them into several broad areas. Table 2.VIII lists the major areas of development for the LANCHESTER theory of combat into which most of the extensions, for example, listed in Table 2.VII fall. In Table 2.IX, we enumerate various papers that fall into these eight major areas. In Table 2.IX, we give the authors' names and date of the published work for each major (or benchmark) piece of work in these areas. The exact reference to each piece of work may be obtained by consulting the list of references at the end of this chapter.

Thus, we hope that Tables 2.VII through 2.IX, Figure 2.11, and these brief comments will provide a rough idea of how the LANCHESTER theory of combat has developed. In the remaining chapters of this book, we will examine in more detail some of the more important topics on combat modelling and/or analysis.

TABLE 2.VIII. Major Areas of Development for LANCHESTER Theory of Combat

1. Stochastic combat models
2. Optimal fire-distribution strategies
   a. optimal air-war strategies
   b. optimal fire-support strategies
3. Empirical verification
4. Different functional forms for attrition rates
5. Applications to guerrilla warfare
6. Prediction of attrition-rate coefficients
7. Variable attrition-rate coefficients
8. Large-scale, complex planning models

TABLE 2.IX. Development of the Major Areas of the LANCHESTER Theory of Combat.

Stochastic Combat Models

- KOOPMAN (1940's; see MORSE and KIMBALL (1951))
- SNOW (1948)
- G. WEISS (1963)
- KISI and HIROSE (1966)
- SPRINGALL (1968)
- GRUBBS and SHUFORD (1973)
- SHUFORD and GRUBBS (1975)
- BROWN (1955, 1963)
- SMITH (1965)
- HELLMAN (1966)
- CLARK (1969)
- KARR (1974, 1975a, 1975b, 1976)
- WATSON (1976)

Variable Attrition-Rate Coefficients

- KOOPMAN (1940's: see MORSE and KIMBALL (1951))
- H. K. WEISS (1957)
- FARRELL (1970)
- TAYLOR and PARRY (1975)
- TAYLOR and COMSTOCK (1977)
- BONDER (1964)
- TAYLOR† (1971, 1974c)
- TAYLOR and BROWN (1976)

Optimal Fire-Distribution Strategies

A. General

- ISBELL and MARLOW (1956b)
- TAYLOR (1973, 1974a, 1974d, 1975)

b. Optimal Air-War Strategies

- MORSE and KIMBALL (1951)
- GIAMBONI, MENGEL, and DISHINGTON (1951)
- MENGEL (1953, 1954)
- FULKERSON and JOHNSON (1957)
- BERKOVITZ and DRESHER (1959, 1960)
- BRACKEN, FALK, and KARR (1975)
- ANDERSON, BRACKEN, and SCHWARTZ (1975)
- ISAACS (1954, 1955, 1965)
- BELLMAN and DREYFUS (1958)
- BRACKEN (1973)

C. Optimal Fire-Support Strategies

- H. K. WEISS (1957, 1959)
- KAWARA (1973)
- TAYLOR (1974, 1977)
- TAYLOR and BROWN (1978)

Different Functional Forms for Attrition Rates

- PETERSON (1953, 1967)
- BRACKNEY (1959)
- HELMBOLD (1965)

Empirical Verification

- ENGEL (1954)
- H. K. WEISS (1957, 1966)
- HELMBOLD (1961a, 1961b, 1964a, 1964b, 1971)
- WILLARD (1962)

Applications to Guerrilla Warfare

- DEITCHMAN (1962)
- KISI and HIROSE (1966)
- SCHAFFER (1968)

Prediction of Attrition-Rate Coefficients

- BRACKNEY (1959)
- SCHAFFER (1968)
- CLARK (1969)
- KIMBLETON (1971)
- BONDER (1964, 1967, 1970)
- BARFOOT (1969)
- BONDER and FARRELL (1970)

Large-Scale, Complex Planning Models

- GIAMBONI, MENGEL, and DISHINGTON (1951)
- SISKA, GIAMBONI, and LIND (1954)
- BONDER (1964)
- BONDER and FARRELL (1970)
- FARRELL (1975)
- CLARK (1969)
- BONDER and HONIG (1971)
- CHERRY (1975)

---

† Here TAYLOR (1974c) = the third paper published by TAYLOR in 1974.

2.8. A Simple Model of Battle Termination.

For assessing the outcomes of combat engagements between units in war games and simulations, one needs some type of "combat results table" that relates the initial conditions of combat to probable outcomes. The military operations analyst is faced with constructing such a table. Let us recall that the first question that we posed in Section 2.2 about the dynamics of combat between two homogeneous forces was (Q1): "Who will 'win'? Be annihilated?" It turns out that the determination of battle outcome depends on not only the dynamics of combat (i.e. differential equations such as (2.2.1), which model the force-attrition processes) but also the battle-termination rules used.

Of even more interest to the military operations analyst is how the means and tactics for waging war are related to the outcome of battle. Specifically, one desires to have a clear understanding of how force-level and weapon-system performance parameters interact to determine a battle's outcome. What is the tradeoff between quality and quantity of weapon systems? When are two forces of equal strength? All such determinations require the specification of a model for battle termination. We will now consider a simple model of battle termination and briefly study its implications for conditions of force superiority. We had to defer the discussion of battle-outcome determination/prediction (i.e. the answering of questions (Q1) through (Q4) of Section 2.2) until now when we will examine battle-termination modelling.

As H. K. WEISS [98] has emphasized, engagements that continue until one side is wiped out are rare. Although we are well aware that battle termination is a complex random process for which it is by no means certain that force levels are the only significant variables (i.e. the state

variables)[28], we will assume that combat ends when either of two given "breakpoint" force levels is first reached. In Chapter 3 we will discuss the modelling of battle termination more thoroughly. Accordingly, for present purposes, let us define a <u>force-level breakpoint</u> as that point (i.e. force level) at which a unit (either offensive or defensive) can no longer perform its mission during a fire fight. We will assume that when a unit's breakpoint force level (or, simply, its breakpoint) is reached, the unit will "break off" the engagement and leave the enemy force in possession of the field of battle. In other words, we consider that when a unit reaches its breakpoint before the enemy has, that unit has lost the battle.

Thus, the simplest model of battle termination is that battle outcome depends (deterministically) only on the force levels. In other words, we are considering a purely deterministic model of battle termination (with no element of chance). In Chapter 3 we will discuss the modelling of battle termination as a stochastic (or random) process. Let us consider combat between two homogeneous forces (denoted as X and Y) and denote X's breakpoint force level as x_{BP}, with y_{BP} being similarly defined. Hence, for example, the following three conditions hold for a Y victory:

$$Y \text{ wins when } \begin{cases} \text{(C1)} \quad x_f = x_{BP}, \\ \text{(C2)} \quad y_f > y_{BP}, \\ \text{(C3)} \quad x(t) > x_{BP} \text{ and } y(t) > y_{BP} \text{ for } 0 \le t < t_f, \end{cases} \tag{2.8.1}$$

where $x(t)$ and $y(t)$ denote the X and Y force levels at time t, and t_f, $x_f = x(t_f)$, and $y_f = y(t_f)$ denote final values. Let us also write that, for example,

$$x_{BP} = f^X_{BP} x_0 , \qquad (2.8.2)$$

where f^X_{BP} denotes a given fraction of X's initial strength. This breakpoint fraction f^X_{BP} (or, equivalently, the unit's breakpoint) is usually assumed to depend on the tactical posture of the unit, its size, etc. Typical values for a company-sized unit are the following:

$$f^X_{BP} = 0.7 \text{ for an attacking force,}$$

and

$$f^X_{BP} = 0.5 \text{ for a defending force.}$$

For any particular battle (i.e. for particular specified values of attrition-rate coefficients and initial force levels) between two homogeneous forces with assumed fixed-force-level breakpoints, we can <u>always</u>, of course, determine the outcome simply by plotting the decay of the force levels $x(t)$ and $y(t)$ and observing which side first reaches its breakpoint. This approach is, however, a time-consuming procedure, and it does not provide any deep understanding of the dynamics of combat (i.e. how weapon-system capabilities and numbers of forces determine the outcome of battle). It is therefore of interest to have available victory-prediction conditions, which explicitly portray the relationship between these variables (i.e. weapon-system-capability and force-level variables) and the outcome of battle.

Thus, we will give victory-prediction conditions for LANCHESTER's classic combat formulations with fixed-force-level breakpoints. We will state these results without proof; details of their development are given in Chapter 3. In other words, we now will give battle-outcome-prediction results that answer questions (Q1) through (Q4) posed in Section 2.2 above for the two classic models:

(M1) LANCHESTER's equations for modern warfare (2.2.1),

and (M2) LANCHESTER's equations for area fire (2.4.1).

Let us therefore first consider the case in which the combat dynamics are given by LANCHESTER's equations for modern warfare (2.2.1). In this case, Y will win a fixed-force-level-breakpoint battle (in finite time) if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}\left\{\frac{1 - (f_{BP}^Y)^2}{1 - (f_{BP}^X)^2}\right\}} \ . \qquad (2.8.3)$$

When (2.8.3) holds and Y wins, the number of his survivors follows from LANCHESTER's square law (2.2.5) and (2.8.2), and it is given by

$$y_f = y_0\sqrt{1 - \frac{b}{a}\left(\frac{x_0}{y_0}\right)^2 \{1 - (f_{BP}^X)^2\}} \ . \qquad (2.8.4)$$

It is also of interest to compute the winner's total casualties (denoted as y_c^f) and also his fractional loss (denoted as $(f_c^Y)_f$), since these quantities are measures of his "cost" for doing combat and achieving victory. In general for cases with no replacements and no withdrawals, Y's total casualties, denoted as y_c, are given by

$$y_c = y_0 - y \ , \qquad (2.8.5)$$

so that (2.8.4) yields that the victor's losses are given by

$$y_c^f = y_0 \left\{1 - \sqrt{1 - \frac{b}{a}\left(\frac{x_0}{y_0}\right)^2 [1 - (f_{BP}^X)^2]}\right\} \ , \qquad (2.8.6)$$

where y_c^f denotes Y's final casualties at the end of battle at t_f. Similarly, Y's casualty fraction is defined (in such cases of no replacements and withdrawals) by

$$f_c^Y = \frac{y_0 - y}{y_0}, \tag{2.8.7}$$

so that (2.8.6) yields that the victor's fractional loss is given by

$$(f_c^Y)_f = 1 - \sqrt{1 - \frac{b}{a}\left(\frac{x_0}{y_0}\right)^2 \{1 - (f_{BP}^X)^2\}}\ , \tag{2.8.8}$$

where $(f_c^Y)_f$ denotes the final casualty fraction.

We denote the time for X to reach his breakpoint as t_{BP}^X, with t_{BP}^Y being similarly defined. The time t_{BP}^X may be determined by solving the equation

$$x(t_{BP}^X) = x_{BP} = f_{BP}^X x_0\ , \tag{2.8.9}$$

and accordingly we obtain using (2.2.8) that

$$t_{BP}^X = \begin{cases} \dfrac{-1}{\sqrt{ab}} \ln(1 - f_{BP}^X) & \text{for } \dfrac{x_0}{y_0} = \sqrt{\dfrac{a}{b}}\ , \\[2ex] \dfrac{1}{\sqrt{ab}} \ln \left\{ \dfrac{-(x_0/y_0)\ f_{BP}^X + \sqrt{(a/b) - (x_0/y_0)^2\ [1 - (f_{BP}^X)^2]}}{\sqrt{a/b} - (x_0/y_0)} \right\} & \\[2ex] & \text{for } \dfrac{x_0}{y_0} \neq \sqrt{\dfrac{a}{b}} \end{cases} \tag{2.8.10}$$

We can obtain a similar result for t_{BP}^{Y}. Then the victory-prediction condition (2.8.3) follows from requiring that $t_{BP}^{X} < t_{BP}^{Y}$. Since the battle ends upon X's force level reaching his breakpoint [see (2.8.1) above], the time at which the battle ends, t_f, is equal to t_{BP}^{X}. Thus, the time for Y to win such a battle, denoted as t_W^Y, is given by $t_W^Y = t_f$. These results are all summarized in Table 2.X. In summary, the information contained in this table provides the answers to the questions (Q1) through (Q4) posed above in Section 2.2.

These results are particularly significant because they show that the outcome of battle is determined by only three relative factors (and not absolute quantities), even though our combat model (2.2.1) (with battle termination conditions included) contains six independent parameters: namely, a, b, x_0, y_0, f_{BP}^{X}, and f_{BP}^{Y}. In particular the victory-prediction condition (2.8.3) explicitly shows the parametric dependence of battle outcome on various combat factors. We see that the outcome of a fixed-force-level-breakpoint battle depends on three factors:

(F1) the initial force ratio, $u_0 = x_0/y_0$

(F2) relative fire effectiveness, $R = a/b$,

and (F3) a relative breakpoint factor, $B = B(f_{BP}^{X}, f_{BP}^{Y})$,

where

$$B(u,v) = \sqrt{\frac{1 - v^2}{1 - u^2}} \quad .$$

All three factors are relative factors. The first two are simply ratios, invariant for certain types of changes in the absolute battle conditions (namely, the group of similarity transformations, which leaves these ratios unchanged). The relative breakpoint factor has the following properties:

TABLE 2.X. Summary of Battle-Outcome Results for LANCHESTER's Equations for Modern Warfare and Fixed Force-Level Breakpoints

$$Y \text{ will win if and only if } \frac{x_0}{y_0} < \sqrt{\frac{a}{b}\left\{\frac{1-(f_{BP}^{Y})^2}{1-(f_{BP}^{X})^2}\right\}}$$

When Y wins:

(A) winner's survivors,
$$y_f = y_0 \sqrt{1 - \frac{b}{a}\left(\frac{x_0}{y_0}\right)^2 \{1-(f_{BP}^{X})^2\}}$$

(B) winner's fractional loss,
$$(f_c^Y)_f = 1 - \sqrt{1 - \frac{b}{a}\left(\frac{x_0}{y_0}\right)^2 \{1-(f_{BP}^{X})^2\}}$$

(C) duration of battle, $t_f = t_W^Y$ where

$$t_W^Y = \begin{cases} \dfrac{-1}{\sqrt{ab}} \ln(1 - f_{BP}^X) & \text{for } \dfrac{x_0}{y_0} = \sqrt{\dfrac{a}{b}} \\[2ex] \dfrac{1}{\sqrt{ab}} \ln\left\{\dfrac{-(x_0/y_0) f_{BP}^X + \sqrt{(a/b) - (x_0/y_0)^2 [1-(f_{BP}^X)^2]}}{\sqrt{a/b} - (x_0/y_0)}\right\} & \\ & \text{for } \dfrac{x_0}{y_0} \neq \sqrt{\dfrac{a}{b}} \end{cases}$$

(a) $B(u,u) = 1$, (b) $\partial B/\partial u > 0$ for $u > 0$, and (c) $\partial B/\partial v < 0$ for $v > 0$. Hence, $B(u,v) > 1$ for $u, v \geq 0$ if and only if $u > v$. We may then rewrite the victory prediction condition (2.8.3) as

$$Y \text{ will win if and only if } \frac{x_0}{y_0} < B(f_{BP}^X, f_{BP}^Y) \sqrt{\frac{a}{b}} . \qquad (2.8.11)$$

Thus, even though for a fixed-force-level-breakpoint battle, the model (2.2.1) contains six independent parameters (including the two breakpoint fractions), it is only the three relative factors, u_0, R, and B, which determine battle outcome. The relative breakpoint factor $B(f_{BP}^X, f_{BP}^Y)$ explicitly shows the influence of the units' breakpoints on battle outcome. In particular, when $f_{BP}^X = f_{BP}^Y$, the victory-prediction condition (2.8.11) reduces to the force-annihilation-prediction condition given in Proposition 2.2.1. It seems appropriate for us to point out here that although we have been able to generalize Proposition 2.2.1 (i.e. generalize force-annihilation-prediction conditions) to the case of time-dependent attrition-rate coefficients, we have not been able to do so for the victory-prediction condition (2.8.11) for a fixed-force-level-breakpoint battle.

Using the results of Table 2.X, we have constructed Table 2.XI. In this latter table we show the influence of the values taken for the units' breakpoints on the outcome of battle. Parameter values were chosen to be representative of an attack by the X forces against Y. Frequently, one hears in military circles that a three-to-one force ratio is necessary for success in attacking an enemy position. Table 2.XI has been constructed to also examine this rule of thumb. Consequently, we have taken a force ratio of 3.00 (numbers of attackers to defenders) for this examination. Additionally,

TABLE 2.XI. Influence of Breakpoints on the Outcome of Battle for an Attack by X Against Y with Battle Dynamics Given by LANCHESTER's Equations for Modern Warfare.

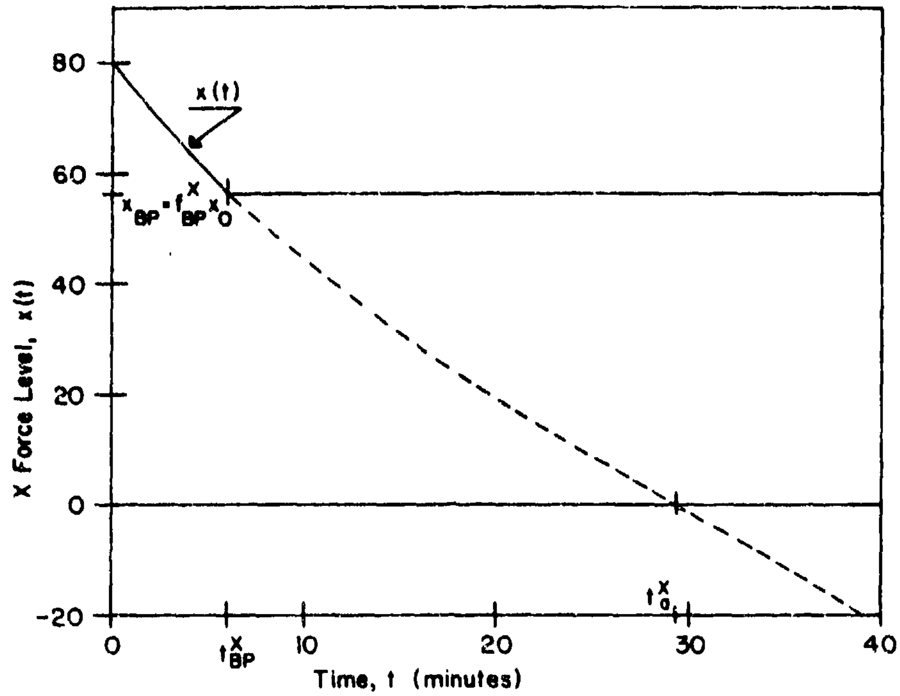
CASE	$\frac{x_0}{y_0}$	$\frac{a}{b}$	f_{BP}^X	f_{BP}^Y	$\sqrt{\frac{a}{b}\left\{\frac{1-(f_{BP}^Y)^2}{1-(f_{BP}^X)^2}\right\}}$	WINNER	$\frac{x_f}{x_0}$	$\frac{y_f}{y_0}$
1	3.0	5.0	0.8	0.5	3.23	Y	0.8	0.59
2	3.0	5.0	0.7	0.5	2.71	X	0.76	0.50
3	3.0	5.0	$f_{BP}^Y = f_{BP}^X$		2.24	X	$> f_{BP}^X$	f_{BP}^Y

NOTE: X is the Attacker.

one would think that the defenders (with their established positions and well-planned "fields of fire") would be relatively more effective (per man) than the attackers. The input values shown in Table 2.XI reflect this situation. Also, the values selected for the two breakpoints, namely f_{BP}^{X} and f_{BP}^{Y}, reflect the hypothesis that the defending unit (which does not move and require as close coordination and control for movement as the attacking unit does) can sustain a higher fraction of casualties than the attacker before abandoning its mission and "breaking off" the engagement.

Let us now examine the sensitivity of battle outcome to the units' breakpoints. If the number contained in column 2 of Table 2.XI is smaller than that contained in column 6, then Y will win according to the above victory-prediction condition (2.8.3). The contents of column 7 (the determined victor) show the sensitivity of battle outcome to the breakpoint values used. Moreover, we should observe that if the battle were to be fought to the annihilation of one side or the other, then X (the attacker) would win. However, since it is usually hypothesized that the attacker can sustain a smaller casualty fraction than the defender before "breaking off" the attack, the attacker may not always win, and the attacker will lose battles for which the "breakpoints overcome mass." For example, X loses the battle identified as Case 1 in Table 2.XI.

Thus, the examples shown in Table 2.XI tell us that a force may be able to win a fixed-force-level-breakpoint battle for certain breakpoints, even though it would lose a fight-to-the-finish. Figure 2.12 shows the decay of the force levels in the more "usual" case in which Y wins with $x_0/y_0 < \sqrt{a/b}$, i.e. X would be annihilated if the battle were allowed to proceed until the annihilation of one side or the other. We have also



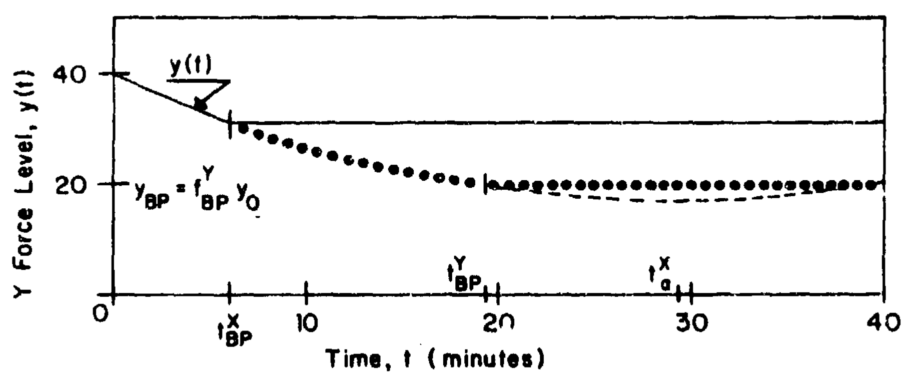


Figure 2.12. Decay of the force levels for LANCHESTER's equations of modern warfare for $0 \le t \le t^Y_W = t^X_{BP} < t^Y_{BP}$ in the case of a Y victory when $x_0/y_0 < \sqrt{a/b}$. The dotted line shows what the decay of the Y force level would be if the forces did not disengage at $t = t^X_{BP}$. The dashed lines extend the X and Y force levels computed according to (2.2.13) and (2.2.14), respectively.

extended, for example, the X force level [computed according to (2.2.13)] past the unit's breakpoint at t_{BP}^{X} and denote this extended curve with a dashed line.

From Figure 2.12 we see that the Y force level [computed according to (2.2.14)] actually increases for $t > t_{a}^{X}$. This should warn the reader against indiscriminate "plugging in" to an equation like (2.2.14). In other words, the attrition equations (2.2.1) are only valid for $x > x_{BP}$ and $y > y_{BP}$. To be precise then, once we have introduced the concept of breakpoints and consider a fixed-force-level-breakpoint battle; we should, for example, write LANCHESTER's equations for modern warfare as

$$\left\{\begin{aligned} \frac{dx}{dt} &= \begin{cases} -ay & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise,} \end{cases} \\[2ex] \frac{dy}{dt} &= \begin{cases} -bx & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}\right. \tag{2.8.12}$$

However, for simplicity we will usually not write out the range of validity of such equations as above and hope that the reader will understand this implied restriction. Figure 2.13 shows that a force that would otherwise be annihilated can actually win a fixed-force-level-breakpoint battle. This situation corresponds to Case 1 shown in Table 2.XI.

Results are similarly obtained when the combat dynamics are given by LANCHESTER's equations for area fire (2.4.1). In this case Y <u>will win a fixed-force-level-breakpoint battle if and only if</u>

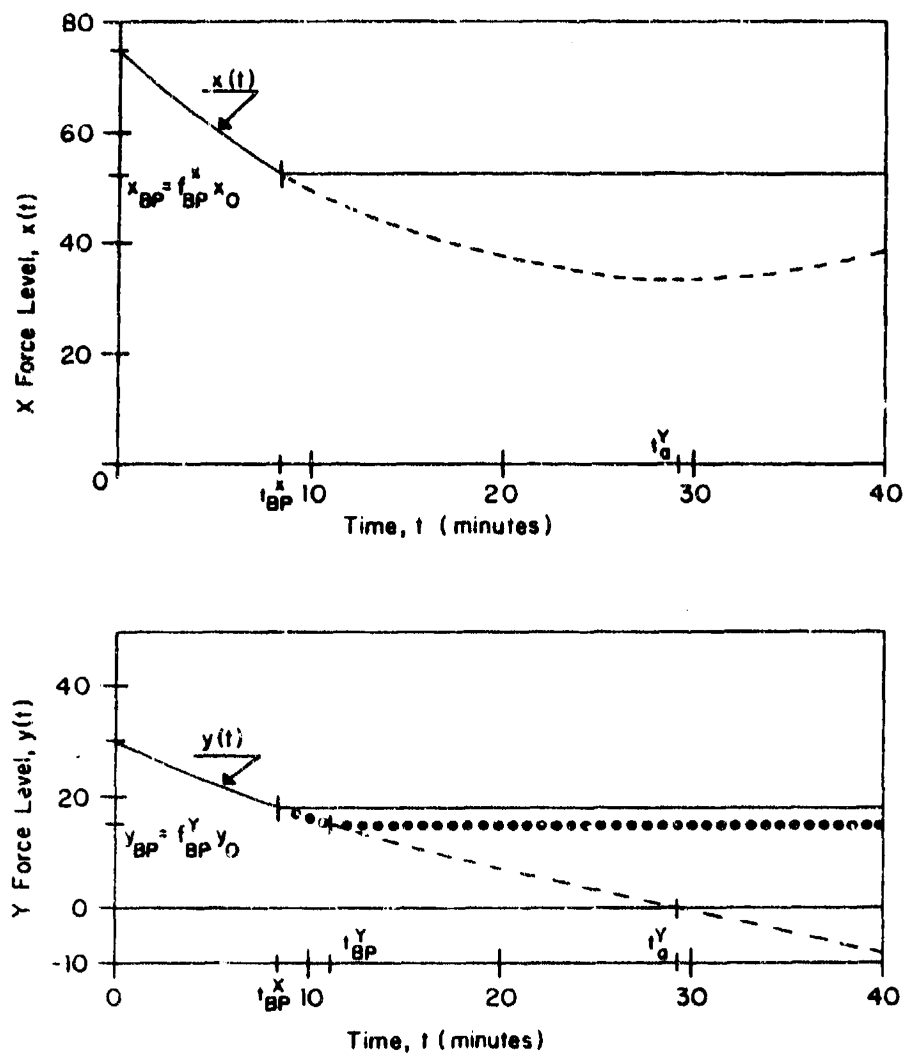


Figure 2.13. Decay of the force levels for LANCHESTER's equations of modern warfare for $0 \le t \le t^Y_W = t^X_{BP} < t^Y_{BP}$ in the case of a Y victory when $x_0/y_0 > \sqrt{a/b}$. The dotted line shows what the decay of the Y force level would be if the forces did not disengage at $t = t^X_{BP}$. The dashed lines extend the X and Y force levels computed according to (2.2.13) and (2.2.14), respectively.

$$\frac{x_0}{y_0} < \frac{a}{b}\left\{\frac{1 - f_{BP}^Y}{1 - f_{BP}^X}\right\} . \qquad (2.8.13)$$

The length of battle is finite, however, if and only if $f_{BP}^X > 0$. Other results are obtained by means similar to those employed in the previous case (i.e. for LANCHESTER's equations for modern warfare). These results are summarized in Table 2.XII.

From the victory-prediction condition (2.8.13) for combat modelled by LANCHESTER's equations for area fire, we again explicitly see the parametric dependence of battle outcome on only three <u>relative</u> combat factors, even though our combat model (2.4.1) (with battle termination conditions included) contains six independent parameters. Although the functional dependence in the victory-prediction condition is different from that for LANCHESTER's equations for modern warfare, we again encounter the same three factors that determine battle outcome: namely, (F1) the initial force ratio, $u_0 = x_0/y_0$, (F2) relative fire effectiveness, $R = a/b$, and (F3) a relative breakpoint factor $B = B(f_{BP}^X, f_{BP}^Y) = (1 - f_{BP}^Y)/(1 - f_{BP}^X)$. The relative breakpoint factor, however, is different for the two different combat dynamics [namely, for combat modelled by (2.2.1) and also (2.4.1)]. For the combat dynamics (2.4.1) the victory-prediction condition takes a particularly simple form in terms of the breakpoint casualty fractions, denoted as $(f_c^X)_{BP}$ and $(f_c^Y)_{BP}$ [<u>see</u> (2.8.7) above]. Thus, Y <u>will win a fixed-force-level-breakpoint battle if and only if</u>

$$\frac{x_0}{y_0} < \frac{a}{b}\,\frac{(f_c^Y)_{BP}}{(f_c^X)_{BP}} , \qquad (2.8.14)$$

or, equivalently,

TABLE 2.XII. Summary of Battle-Outcome Results for LANCHESTER's Equations for Area Fire and Fixed Force-Level Breakpoints

$$Y \text{ will win if and only if } \frac{x_0}{y_0} < \frac{a}{b}\left\{\frac{1 - f_{BP}^{Y}}{1 - f_{BP}^{X}}\right\}$$

When Y wins:

(A) winner's survivors,
$$y_f = y_0\left\{1 - \frac{b}{a}\left(\frac{x_0}{y_0}\right)(1 - f_{BP}^{X})\right\}$$

(B) winner's fractional loss, $(f_c^Y)_f = \frac{b}{a}\left(\frac{x_0}{y_0}\right)(1 - f_{BP}^{X})$

(C) duration of battle, $t_f = t_W^Y$ where

$$t_W^Y = \begin{cases} \frac{1}{ay_0}\left(\frac{1}{f_{BP}^{X}} - 1\right) & \text{for } \rho = 1 \\[2ex] \frac{1}{ay_0(1-\rho)}\ln\left\{\rho + \left(\frac{1-\rho}{f_{BP}^{X}}\right)\right\} & \text{for } \rho \neq 1 \end{cases}$$

and
$$\rho = \frac{b}{a}\left(\frac{x_0}{y_0}\right) .$$

NOTE: X will win when $0 \le f_{BP}^{X} < 1 - \frac{1}{\rho}$.

$$\left(\frac{x_0}{y_0}\right)\left(\frac{a}{b}\right)\left\{\frac{(f_c^X)_{BP}}{(f_c^Y)_{BP}}\right\} < 1 . \qquad (2.8.15)$$

In (2.8.15) the victory-prediction condition is expressed in terms of the product of three relative factors, each the ratio of the X quantity to that for Y). Let us stress that it is only for LANCHESTER's equations for area fire (2.4.1) for which such simple results are possible. This is even more true when each side's breakpoint is considered to be a random variable (see Chapter 3).

Let us finally discuss some of the differences between the above results for LANCHESTER's equations for modern warfare (2.2.1) and those for LANCHESTER's equations for area fire (2.4.1). It seems appropriate to say that two forces are of equal fighting strength for a particular battle if neither force will win, i.e. either (O1) neither side's breakpoint is ever reached, or (O2) both are reached simultaneously. Table 2.XIII then gives the conditions for equality of fighting strengths for the two attrition models (2.2.1) and (2.4.1). From this table we see that equality of fighting strengths not only depends on the battle-termination conditions but also in different ways for the two models. Such parity conditions may be considered to provide a tradeoff between the quantity and the quality of weapon systems.

Furthermore, Table 2.XIII shows us that such quantity-quality tradeoffs are quite different for these two classic combat attrition models. For LANCHESTER's equations of modern warfare, a four-fold increase in the relative effectiveness of enemy (for example, Y) weapons can be offset by a doubling in the ratio of friendly to enemy forces (i.e. increasing a/b

TABLE 2.XIII. Conditions for Equality of Fighting Strengths in a Fixed-Force-Level-Breakpoint Battle for LANCHESTER's Two Class Models.

(M1) LANCHESTER's Equations for Modern Warfare

$$\frac{x_0}{y_0} = \left\{ \sqrt{\frac{1 - (f_{BP}^{Y})^2}{1 - (f_{BP}^{X})^2}} \right\} \sqrt{\frac{a}{b}}$$

(M2) LANCHESTER's Equations for Area Fire

$$\frac{x_0}{y_0} = \left\{ \frac{1 - f_{BP}^{Y}}{1 - f_{BP}^{X}} \right\} \frac{a}{b}$$

by a factor of four can be offset by increasing x_0/y_0 by a factor of two). In a sense then, increasing the number of weapons for a side is much more effective in maintaining military parity between two forces than increasing their relative quality. However, for LANCHESTER's equations of area fire (or, for that matter, any "linear-law" attrition process [see Section 2.9 below]) the trading of numbers for quality is "one for one," i.e. a four-fold increase in the relative effectiveness of enemy weapons can be offset by a four-fold increase in the ratio of friendly to enemy forces.

Finally, let us remark that the significant thing is that the battle-termination model *is* important and not so much that there is thus and so a functional relationship between parameters of the battle-termination model and the force-parity condition. The actual real-world process of battle termination is much more complicated than the simple model considered here. Thus, the most significant aspect of our work here is the fact that battle termination *must* be considered in determining force parity.

2.9. Concentration of Forces Revisited.

One of the half dozen or so principles of war is the principle of concentration (or mass), which would have a commander concentrate as many men and means as possible at the decisive point in battle. As we have seen above in Section 2.1, F. W. LANCHESTER sought to develop in his now classic 1914 paper, a quantitative justification for the principle of concentration with an idealized model of the combat process. We will now examine this topic in more depth than in Section 2.1, however. LANCHESTER [55, p. 422, column 1] points out that there are two aspects of the principle of concentration: (1) mental concentration (i.e. focusing all mental energy on a single objective), and (2) material concentration (i.e. focusing all material means on a single objective). He will focus on the second aspect of the principle of concentration (i.e. material concentration), however.

In other words, LANCHESTER hypothesizes that in "modern" warfare there are substantial benefits to be gained from merely committing more forces to battle. He will seek to investigate the underlying principles that cause such "economies of scale" in combat. As we have seen above, his models of combat (2.2.1) and (2.4.1) were the result of this investigation. Not only did LANCHESTER show that there were increasing returns to scale from committing additional force to battle, but he also developed an important tradeoff for quality versus quantity of weapon systems by means of his famous square law, namely, the condition for equality of "fighting strength"

$$\left(\frac{x_0}{y_0}\right)^2 = \frac{a}{b} \tag{2.9.1}$$

Before going further, however, let us point out an important distinction between the sense in which we and LANCHESTER use the term "concentration" and that used by most military analysts today.[29] Today the term "concentration" of forces is usually used in the context of a single force split into two or more subunits for purposes of massing and/or economy of force. In this sense, one must consider the cost to the parent unit of concentration of forces in one sector at the expense of another sector. As COLONEL VASILIY Y. SAVKIN [71] of the USSR has stated,

> "To attain victory over the enemy one must not dissipate his forces and means equally across the entire front, but the main efforts must be concentrated on the most important axis or sector and at the right time in order to form there the necessary superiority over the enemy in men and weapons."

We will not use the term in this more sophisticated sense, but we will consider only one battle and will examine the consequences of initially committing additional forces to combat.

Let us now address the question, "What are the benefits to be gained from committing additional forces to battle?" Our problem is to model and evaluate the consequences of this action. We have given this question a cursory examination in Section 2.1 above, and we will examine it in more depth here. In particular, we will contrast results for the two models (2.2.1) and (2.4.1).

Let us now consider the question of whether or not to commit additional forces to battle as a decision problem faced by one of two commanders about to engage in combat. Without loss of generality, we may play the role of the Y commander. Our problem is to find the "best" value for the initial number of forces committed to battle by Y, denoted as y_0. In other words, y_0 is the decision variable for Y in our decision problem. Let us now ask ourselves what are the factors affecting Y's decision. The main factors affecting Y's initial commitment of forces appear to be:

(F1) what the Y commander knows about the battlefield situation,

(F2) what the enemy commander (i.e. X) will decide to do,

(F3) nature of the combat attrition processes,

(F4) criterion selected by Y for evaluating the consequences of his action,

(F5) how the battle will be terminated,

(F6) who will win the battle,

(F7) subsequent combat actions.

For simplicity, we will ignore the last factor (F7) and consider only the battle at hand. Let us consider the case in which Y will be the victor (i.e. assume that he has more than enough forces available to "win" the battle). We will then consider the initial-commitment decision by Y as a one-sided optimization problem: we assume that the X-force commander had adopted a known course of action and consider Y's initial-commitment decision in this light.

Based on the above consideration, the essential aspects of the decision process for Y in deciding whether or not to concentrate forces

(i.e. initially commit as many as possible to battle) are the following:

(1) action to be taken (decision variable),

(2) information available to decision maker,

(3) outcome "yardstick" (decision criterion),

(4) relationship of action to outcome (system dynamics and nature of planning horizon).

In our initial investigation here let us not consider the inherent uncertainty in the decision problems and assume that Y has perfect knowledge about x_0 and y_0, the battle dynamics (assumed deterministic) and battle termination (also assumed deterministic).[30] Hence, we will not consider the information structure here further, although it will certainly play a major role in actual real-world military decisions. Let us summarize, our assumptions about our decision problem:

(A1) enemy (i.e. X's) course of action fixed,

(A2) nature of battle dynamics remains the same during the engagement,

(A3) Y has more than enough forces to "win" the battle and additional forces can be committed to battle in any quantity desired,

(A4) Y knows the numerical strengths and capability of each side,

(A5) battle will be terminated by a fixed force-level breakpoint force level being reached.

As we have discussed in Chapter 1 above, one of the major decisions in evaluating any system or operation is the selection of the appropriate evaluation criteria or measures of effectiveness. For our idealized concentration-of-forces decision, we will consider a single measure of effectiveness (MOE). We are assuming that Y has more than enough forces available to "win" the battle, so therefore Y will always wind up in

sole possession of the battlefield. It therefore seems appropriate to take some measure of the cost of achieving this victory as the criterion for deciding whether or not it will be worthwhile to commit additional forces to battle. A natural measure of the "cost of doing battle" is the number of casualties sustained by the Y force. Let us denote the number of casualties as y_c. We have then that $y_c = y_f - y_0$, where y_f denotes the final Y force level at the end of battle when the X breakpoint (denoted as x_{BP}) has been reached. We also have then $x_f = x_{BP} = f^X_{BP} x_0$. Let us note that since the battle is terminated by X reaching his breakpoint, X's casualties are always the same [namely, $x_c = x_f - x_0 = (1 - f^X_{BP})$], regardless of how many forces Y initially commits to battle.

Thus, we may state in quantitative terms the decision problem of determining the "best" initial commitment of Y's forces as

$$\underset{y_0}{\text{minimize}} \; C, \qquad \text{subject to: } y_0^{\min} \leq y_0 \leq y_0^{\max}, \tag{2.9.2}$$

where $C = y_c = y_f - y_0$ denotes the cost of doing combat (i.e. the decision criterion or objective function), y_0 is the decision variable for which the best (i.e. optimal) value is to be determined, $y_0^{\min} = y_0^{draw} + \varepsilon$, $\varepsilon > 0$, and y_0^{draw} denotes the value of the initial Y force level that leads to a draw in a fixed-force-level-breakpoint battle. We will denote the optimal value of y_0 as y_0^*. We have now specified all aspects of our combat optimization problem except for the combat dynamics. We will consider the above combat-optimization problem (2.9.2) for two classes of battle dynamics:

(C1) "square-law" battles, and

(C2) "linear-law" battles.

By a "square-law battle" we mean any LANCHESTER-type battle for which LANCHESTER's square-law,

$$b\{x_0^2 - x^2(t)\} = a\{y_0^2 - y^2(t)\}, \tag{2.9.3}$$

holds as the state equation. It follows that the combat dynamics must be given by

$$\begin{cases} \dfrac{dx}{dt} = -ay\cdot\gamma(t,x,y) \ , \\ \\ \dfrac{dy}{dt} = -bx\cdot\gamma(t,x,y) \ . \end{cases} \tag{2.9.4}$$

To insure a militarily realistic situation in which both dx/dt and $dy/dt \leq 0$ for $x, y \geq 0$, we assume that $\gamma(t,x,y) \geq 0$ for $x, y \geq 0$ and all t. Lanchester's equations for modern warfare (2.2.1) are, of course, an example of such battle dynamics. However, any battle for which (2.9.3) holds will yield the same results as far as concentration of forces is concerned, and this is why we consider the more general combat dynamics (2.9.4). To insure that the battle terminates in finite time, we assume that

$$\lim_{T \to +\infty} \int_0^T \gamma(t, x(t), y(t)) = +\infty \ . \tag{2.9.5}$$

Results for a "square-law" battle are shown in Table 2.XIV. In this case, Y will win a fixed-force-level-breakpoint battle (in finite time) if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}\left\{\frac{1 - (f_{BP}^{Y})^2}{1 - (f_{BP}^{X})^2}\right\}} \ .$$

TABLE 2.XIV

Variation in Own Casualties for Changes in Initial Number of Own Forces in "Square-Law" Battle with Fixed Force-Level Breakpoint

Combat Dynamics:
$$\begin{cases} \dfrac{dx}{dt} = -ay\cdot\gamma(t,x,y) \\ \dfrac{dy}{dt} = -bx\cdot\gamma(t,x,y) \end{cases}$$

For $\dfrac{x_0}{y_0} < \sqrt{\dfrac{a}{b}\left\{\dfrac{1-(f_{BP}^{Y})^2}{1-(f_{BP}^{X})^2}\right\}}$,

A. Battle Outcome: Y wins with
$$\begin{cases} y_f = y_0\sqrt{1-\dfrac{b}{a}\left(\dfrac{x_0}{y_0}\right)^2\{1-(f_{BP}^{X})^2\}} \\ x_f = f_{BP}^{X}x_0 \end{cases}$$

B. Own Casualties, $y_c = y_0 - y_f$:

$$y_c = y_0\left\{1-\sqrt{1-\frac{b}{a}\left(\frac{x_0}{y_0}\right)^2[1-(f_{BP}^{X})^2]}\right\}$$

$$\frac{\partial y_c}{\partial y_0} = -\left(\frac{y_0-y_f}{y_f}\right) < 0$$

$$\frac{\partial^2 y_c}{\partial y_0^2} = \frac{1}{y_f}\left\{\left(\frac{y_0}{y_f}\right)^2 - 1\right\} > 0$$

When Y wins, the cost of doing battle, namely $C = y_c$, is given by

$$C = y_0 \left\{ 1 - \sqrt{1 - \frac{b}{a}\left(\frac{x_0}{y_0}\right)^2 [1 - (f_{BP}^X)^2]} \right\} \qquad (2.9.6)$$

Since $\partial C/\partial y_0 < 0$ always (see Table 2.9.1), $y_0^* = y_0^{max}$, and the victory Y should always initially commit as many forces as possible to battle, regardless of what the breakpoints are (as long as Y will win). Furthermore, $\partial^2 C/\partial y_0^2 > 0$ so there are diminishing returns from committing additional forces to battle. Thus, irrespective of what the breakpoints are (as long as Y will win), Y should always initially commit as many forces as possible to battle when combat attrition yields Lanchester's square law (2.9.3). The reader should recall that in Section 2.1 we said that Lanchester's square law (2.1.5) yields the important implication that a side can always significantly reduce its own casualties by initially committing additional forces to battle. However, we did not prove the validity of this assertion earlier but merely contented ourselves with a numerical demonstration of its plausibility.

Similarly, by a "linear-law battle" we mean any Lanchester-type battle for which Lanchester's linear law,

$$b\{x_0 - x(t)\} = a\{y_0 - y(t)\}, \qquad (2.9.6)$$

holds as the state equation. It follows that the combat dynamics must be given by

$$\frac{dx}{dt} = -a \cdot \mu(t,x,y) \ ,$$
$$\frac{dy}{dt} = -b \cdot \mu(t,x,y) \ . \qquad (2.9.7)$$

Again we assume that $\mu(t,x,y) \geq 0$ so that both dx/dt and $dy/dt \leq 0$. Lanchester postulated (see Section 2.1 above) that such equations held for ancient warfare. He also postulated that another case was that for "area" fire [see equation (2.4.1)]. To insure that one side or the other is eventually annihilated, we assume that

$$\lim_{T \to +\infty} \int_0^T \mu(t,x(t),y(t))\, dt \geq \begin{cases} \dfrac{y_0}{b} & \text{for} \quad \dfrac{x_0}{y_0} \geq \dfrac{a}{b}, \\[2ex] \dfrac{x_0}{a} & \text{for} \quad \dfrac{x_0}{y_0} \leq \dfrac{a}{b}. \end{cases} \tag{2.9.8}$$

Results for a "linear-law" battle are shown in Table 2.XV. In this case, Y will win a fixed-force-level-breakpoint battle if and only if

$$\frac{x_0}{y_0} < \frac{a}{b}\left\{\frac{1 - f_{BP}^{Y}}{1 - f_{BP}^{X}}\right\}.$$

When Y wins, the cost of doing combat, namely, $C = y_c$, is given by

$$C = \frac{b}{a}\,[1 - f_{BP}^{X}]x_0\,. \tag{2.9.9}$$

Since $\partial C/\partial y_0 \equiv 0$ (see also Table 2.XV), C does not depend on y_0 at all, so that the cost of doing combat is not affected by varying the initial number of friendly forces committed to battle. In other words, there is no advantage to be gained from concentration of forces in a "linear-law" battle.

Thus, we have shown that the victor's decision as to whether or not to concentrate forces in a fixed-force-level-breakpoint battle for

TABLE 2.XV

Variation in Own Casualties for Change in Initial Number of Own Forces in "Linear-Law" Battle with Fixed Force-Level Breakpoint

Combat Dynamics:
$$\begin{cases} \dfrac{dx}{dt} = -a\cdot\psi(t,x,y) \\ \dfrac{dy}{dt} = -b\cdot\psi(t,x,y) \end{cases}$$

For $\dfrac{x_0}{y_0} < \dfrac{a}{b}\left\{\dfrac{1 - f_{BP}^{Y}}{1 - f_{BP}^{X}}\right\}$,

A. Battle Outcome: Y wins with
$$\begin{cases} y_f = y_0\left\{1 - \dfrac{b}{a}\left(\dfrac{x_0}{y_0}\right)[1 - f_{BP}^{X}]\right\} \\ x_f = f_{BP}^{X}\, x_0 \end{cases}$$

B. Own Casualties, $y_c = y_0 - y_f$:

$$y_c = \frac{b}{a}[1 - f_{BP}^{X}]x_0$$

$$\frac{\partial y_c}{\partial y_0} = 0$$

$$\frac{\partial^2 y_c}{\partial y_0^2} = 0$$

which (A1) through (A5) hold is fundamentally different for square-law battles and for linear-law battles: in a square-law battle it is always best for the victor to initially commit as many forces as possible to combat, while in a linear law battle there is no benefit to be gained from concentrating forces. If we assume (as Lanchester did) that warfare in ancient times consisted of linear-law battles (in which "weapon directly answered weapon" in one-on-one duels) while under modern conditions it consists of square-law battles (in which fire from many may be concentrated on a few), then we see that the importance of concentrating forces has changed appreciably from ancient times to modern times. Under modern conditions, there is then a tremendous advantage to concentrating forces (or at least Lanchester hypothesized so[31]). These results are independent of the breakpoints of both sides (as long as the outcome is not changed). They also hold for any decision criterion, C, which is of the form $C = F(y_0)$, where $F(v)$ is a strictly increasing function of its argument v.

In general, we would want to include enemy casualties[32] in Y's force-concentration decision. However, if we had considered a decision criterion of the form $C = G(x_c, y_c)$, where $G(u,v)$ is a strictly increasing function of its second argument for any fixed value of its first argument u, then we would have reached the same force-concentration decisions as above (e.g. vector should concentrate forces in square-law battle), since $x_c = x_0 = x_{BP} =$ CONSTANT. Thus, one would make the same force-concentration decision for other criteria (i.e. measures of effectiveness) such as the loss difference, $D_c = y_c - x_c$, or the loss ratio, $R_c = y_c/x_c$. Later on, we see that such insensitivity to changes in the decision criterion is due to the battle termination rule (fixed force-load breakpoint), and in other cases the force-concentration decision may depend on the decision criterion (<u>see</u> also TAYLOR [91]).

There is, however, a very simple principle that underlies all the above concentration-of-forces results: namely, the instantaneous casualty-exchange ratio determines the overall casualty-exchange ratio and related measures of relative casualty-production effectiveness; in particular, if the instantaneous casualty-exchange ratio (friendly to enemy) always decreases as the force ratio (enemy to friendly) decreases, then additional forces should be committed to battle by the victor (friendly forces). Let us heuristically show why the latter decision rule for initially committing additional forces to battle is optimal. The key point is that we should think of the instantaneous casualty-exchange ratio dy/dx, as the "cost" to Y of reducing the X force level a unit amount. Thus,

$$\frac{dy}{dx} = \begin{pmatrix} \text{instantaneous} \\ \text{casualty-exchange} \\ \text{ratio} \end{pmatrix} = \begin{pmatrix} \text{"cost" to Y of} \\ \text{reducing X force level} \\ \text{a unit amount} \end{pmatrix} \qquad (2.9.10)$$

Next, we observe that if Y initially commits more forces to battle, then the battle is fought at lower force ratios (regardless of the breakpoints of the two forces). Here we take the force ratio to be the ratio of the enemy (i.e. X) force level to the friendly force level. In other words, we have that the force ratio, u, is given by $u = x/y$. What happens to the instantaneous casualty-exchange ratio if the battle is fought at lower force ratios? The answer to this question may be obtained by considering the following partial derivative

$$\frac{\partial}{\partial u}\left(\frac{dy}{dx}\right), \qquad (2.9.11)$$

which tells us how the instantaneous casualty-exchange ratio varies as the force ratio changes. A positive value for this partial derivative (2.9.11) means that the instantaneous casualty-exchange ratio decreases as the force ratio decreases. It follows that if $(\partial/\partial u)(dy/dx) > 0$ always, then the Y force (whom we assume will win) can reduce the "cost" of doing combat by initially committing more forces to battle and fighting the battle at lower force ratios with their more favorable exchange ratios.

For the "square-law" battle (2.9.4), we have

$$\frac{dy}{dx} = \frac{bx}{ay} = \frac{b}{a} u , \tag{2.9.12}$$

where $u = x/y$, and hence

$$\frac{\partial}{\partial u} \left(\frac{dy}{dx}\right) = \frac{b}{a} > 0 . \tag{2.9.13}$$

It is this result (2.9.13) that explains why it is always a good tactic for Y to concentrate forces (i.e. make y_0 as large as possible in square-law battles). For the "linear-law" battle (2.9.7), however, we have

$$\frac{dy}{dx} = \frac{b}{a} , \tag{2.9.14}$$

so that

$$\frac{\partial}{\partial u} \left(\frac{dy}{dx}\right) \equiv 0 . \tag{2.9.15}$$

In this latter case, therefore, the instantaneous casualty-exchange ratio cannot be changed by varying the force ratio. Hence, the overall casualty-exchange ratio cannot be changed by committing more forces to battle, and there is no advantage to concentrating forces in a fixed force-level breakpoint battle. In Chapter 8 we will rigorously prove such statements in general for Lanchester-type combat with two force-level variables.

Some final reflections seem to be in order. Our heuristic explanation of the underlying reason for wanting to concentrate forces in square-law battles (namely, to reduce the instantaneous casualty-exchange ratio) has shown us that the instantaneous casualty-exchange ratio conveys the basic nature of the casualty-exchange process. We immediately know (without having to explicitly determine any type of state equation) the sensitivity of the overall casualty-exchange ratio and related measures to variations in the initial number of forces committed to battle by determining this key quantity (namely, the instantaneous casualty-exchange ratio), and its sensitivity to force-level changes. Thus, important information about the behavior of our combat model has been obtained without having to spend the time and effort to explicitly compute force-level trajectories.

*2.10. FISKE-Type Equations of Warfare.

H. K. WEISS [101] has pointed out that LANCHESTER, an Englishman, was anticipated (in qualitative but not quantitative terms) in 1905 by BRADLEY A. FISKE (then Commander but later Rear Admiral, USN), an American. FISKE won the Naval Institute Prize for 1905 for his essay entitled "American Naval Policy." In this work he considered a "fire fight" between two fleets (i.e. shots being exchanged between the two fleets within effective gun range of each other) and assumed that both the strengths of the forces and damages sustained could be given numeral values.[33] FISKE then assumed that the damage done to one force by the other, in a given time period, was proportional to the value of the opposing force at the beginning of the time period. He then developed tables to show "how the values of two contending forces change as the fight goes on." He found that the decrease in offensive power of a weaker fleet, fighting a stronger, is geometrical (instead of arithmetical) and that there is a continually increasing difference between the powers of the two fleets as an action that favors the stronger fleet) progresses. Although no equations were given, it is clear that FISKE had gone through all of the logical development for the model (2.2.1).

J. ENGEL [25] subsequently pointed out that FISKE's verbal model is equivalent to a system of difference equations. Let us accordingly consider combat between two homogeneous forces in which casualties are assumed at discrete points in time. We may think of the engagement as being fought in distinct volleys (i.e. discrete exchanges of fire).

Assuming that casualties during a time period are proportional to the number of energy firers at the beginning of the time period, we find that the general equations of FISKE's model are[34]

$$\begin{cases} x_{n+1} - x_n = -\alpha y_n & \text{with } x_{n=0} = x_0 , \\ y_{n+1} - y_n = -\beta x_n & \text{with } y_{n=0} = y_0 , \end{cases} \tag{2.10.1}$$

where the subscript n denotes the nth time period (i.e. just after the nth volley), the battle begins at $n = 0$, x_n and y_n denote the numbers of X and Y combatants that are effective at the beginning of the nth time period, and α and β are positive constants that represent the effectiveness of each side's fire. For example, α denotes the number of X casualties produced by a single Y firer during one time period. Let us refer to the above equations (2.10.1) as <u>FISKE's equations for modern warfare</u>. They are the discrete analogue of LANCHESTER's equations for modern warfare (2.1.1). The relationship between these two models is examined more closely in Appendix E. Intuitively, we would expect the model (2.2.1) and all associated results to be the limiting case of (2.10.1) as the time between volleys becomes arbitrarily small. For now, however, we will briefly examine some of the principal properties and results for the model, with their development deferred until Section 7.5 below.

The X <u>force level</u> at the beginning of time period n, x_n, is given by

$$x_n = \frac{1}{2}\left\{\left(x_0 - y_0\sqrt{\frac{\alpha}{\beta}}\right)\left[1 + \sqrt{\alpha\beta}\right]^n + \left(x_0 + y_0\sqrt{\frac{\alpha}{\beta}}\right)\left[1 - \sqrt{\alpha\beta}\right]^n\right\}, \tag{2.10.2}$$

and similarly for the Y force level

$$y_n = \frac{1}{2}\left\{\left(y_0 - x_0\sqrt{\frac{\beta}{\alpha}}\right)\left[1 + \sqrt{\alpha\beta}\right]^n + \left(y_0 + x_0\sqrt{\frac{\beta}{\alpha}}\right)\left[1 - \sqrt{\alpha\beta}\right]^n\right\} \quad , \tag{2.10.3}$$

Let us assume that[35] $\alpha\beta < 1$ so that $1 - \sqrt{\alpha\beta} > 0$. Similar to the proof of Proposition 2.2.1, it follows that only one of x_n and y_n can ever become negative (i.e. if $x_N < 0$ for some N, then $y_n > 0$ for all $n \geq 0$). Since $[1 + \sqrt{\alpha\beta}]^n > 0$ and $\to +\infty$ as $n \to \infty$ and $[1 - \sqrt{\alpha\beta}]^n > 0$ and $\to 0$ as $n \to \infty$, it follows from (2.10.2) that the following proposition holds.

PROPOSITION 2.10.1: Y will win a fight-to-the-finish in finite time if and only if $x_0/y_0 < \sqrt{\alpha/\beta}$.

Furthermore, it follows from (2.10.1) that

$$\beta\{x_{n+1}^2 - (1-\alpha\beta)x_n^2\} = \alpha\{y_{n+1}^2 - (1-\alpha\beta)y^2\}, \tag{2.10.4}$$

from which we obtain the <u>discrete-time state equation for FISKE's model of modern warfare</u>.

$$\beta\{x_n^2 - (1-\alpha\beta)^n x_0^2\} = \alpha\{y_n^2 - (1-\alpha\beta)^n y_0^2\} \tag{2.10.5}$$

We will also refer to (2.10.5) as <u>FISKE's square law</u>. Let us observe that FISKE's square law (2.10.5) is somewhat different than LANCHESTER's square law (2.2.1) because of the "time-dependent" factor $(1 - \alpha\beta)^n$. However,

the parametric dependence of force annihilation (compare Propositions 2.2.1 and 2.10.1) is exactly the same. In fact most of the solution properties of (2.2.1) and (2.10.1) and their implications are exactly the same.

From the above and results given in Sections 2.2 and 2.3, we see that the models of LANCHESTER and FISKE exhibit the same general behavior. Thus, it has not been critical whether we model time as being continuous or discrete. It is reassuring that the representation of time in our combat model is not the significant feature, but rather the functional relationship for casualty trading is the underlying significant feature. Our model possesses a basic type of invariance that does not depend on the representation of time. Many scientists believe that such invariance is the most significant aspect of many physical laws.[36]

2.11. Comparison of LANCHESTER's Two Basic Models and Summary.

In this section we collect and compare results for LANCHESTER's two classic combat models, i.e. his equations for modern warfare and those for area fire (see Table 2.XVI), with each other. For convenience and also reasons of historical precedence, we have, for example, referred to (2.2.1) as simply LANCHESTER's equations of modern warfare[15], although, of course, several sets of assumptions have been hypothesized to yield them. In the next section (i.e. Section 2.12), however, we develop a more precise notation for referring to such attrition processes.

In table 2.XVII we give an abbreviated description, denoted as "short form," of two alternative sets of assumptions that have been hypothesized to yield each of LANCHESTER's two classic combat models. A more thorough enumeration, denoted as "long form," of the first set of these assumptions is given in each of Tables 2.XVIII and 2.XIX for each of these two basic and classic combat models. The reader should observe in Tables 2.XVIII and 2.XIX that the three assumptions above the dotted line are the same for each model. Also, we have given explicit expressions for the attrition-rate coefficients in each model. To keep these expressions simple, we have made assumption (A3), which is not essential for the functional form of these attrition rates (e.g. attrition rate proportional to the number of enemy firers). Also, WEISS [99, pp. 83-84] has pointed out that assumption (A2) can be weakened: the same equations apply when two homogeneous forces are deployed along a front facing each other with uniform troop density on each side provided that (A2) holds within given force boundaries or "cells" on each side of the front.

In Table 2.XVIII, $t_{ac_{XY}}$ denotes the time for a Y firer to acquire an X target. Here the first force subscript, i.e. the X which is the one closest to the left-hand side of the differential equation, refers to

Table 2.XVI. LANCHESTER's Two Basic Combat Models.

LANCHESTER's Equations for Modern Warfare	LANCHESTER's Equations for Area Fire
$\begin{cases} \frac{dx}{dt} = -ay \\ \frac{dy}{dt} = -bx \end{cases}$	$\begin{cases} \frac{dx}{dt} = -axy \\ \frac{dy}{dt} = -bxy \end{cases}$

Table 2.XVII. SHORT FORM of Alternative Conditions Under Which LANCHESTER's Two Basic Combat Models Have Been Hypothesized to Apply.

	LANCHESTER's Equations for Modern Warfare	LANCHESTER's Equations for Area Fire
1. First Alternative Simple Set of Assumptions	(M1) "aimed" fire (M2) time to acquire an enemy target independent of enemy force level (a special case which is that in which target-acquisition time is negligible)	(A1) "area" fire (A2) constant-area defense
2. Second Alternative Simple Set of Assumptions	($\bar{M}$1) "area" fire ($\bar{M}$2) constant-density defense	($\bar{A}$1) "aimed" fire ($\bar{A}$2) time to detect an enemy target inversely proportional to enemy force level and much greater than time to kill an acquired target

Table 2.XVIII. LONG FORM of Conditions Under Which LANCHESTER's Equations for Modern Warfare Have Been Hypothesized to Apply.

EQUATIONS:

$$\begin{cases} \dfrac{dx}{dt} = -ay \quad \text{with} \quad \dfrac{1}{a} = t_{ac_{XY}} + \dfrac{1}{\nu_Y P_{SSK_{XY}}} \\[2em] \dfrac{dy}{dt} = -bx \quad \text{with} \quad \dfrac{1}{b} = t_{ac_{YX}} + \dfrac{1}{\nu_X P_{SSK_{YX}}} \end{cases}$$

ASSUMPTIONS (after H. K. WEISS [99]):

(A1) Two homogeneous forces are engaged in a fire fight. In other words, the units (i.e. weapon systems) on each side are identical (i.e. every unit on a particular side has exactly the same capability for killing enemy forces and also exactly the same vulnerability to enemy action), but the units on one side may have a different kill rate than opposing enemy units.

(A2) Each unit on either side is within weapon range of all units on the other side.

(A3) The effects of successive rounds in the target areas are independent.

(A4) Each unit is sufficiently well aware of the location and condition of all enemy units so that it engages only live enemy units (one at a time) and kills them at a constant rate, which does not depend on the enemy force level. When an enemy target is killed, search begins for a new target, with the rate of acquiring a new enemy target being independent of the enemy's force level.

(A5) Fire is uniformly distributed over surviving enemy units.

NOTE: See text for explanation of notation.

Table 2.XIX. LONG FORM of Conditions Under Which LANCHESTER's Equation for Area Fire Have Been Hypothesized to Apply.

EQUATIONS:

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{a_{V_X}}{A_X}\nu_Y\, P(K|H)_{XY}\, xy \\[2ex] \dfrac{dy}{dt} = -\dfrac{a_{V_Y}}{A_Y}\nu_X\, P(K|H)_{YX}\, xy \end{cases}$$

ASSUMPTIONS (After H. K. WEISS [99]):

(A1) Two homogeneous forces are engaged in a fire fight. In other words, the units (i.e. weapon systems) on each side are identical (i.e. every unit on a particular side has exactly the same capability for killing enemy forces and also exactly the same vulnerability to enemy action), but the units on one side may have a different kill rate than opposing enemy units.

(A2) Each unit on either side is within weapon range of all units on the other side.

(A3) The effects of successive rounds in the target areas are independent.

(A4) Each firing unit is aware only of the general area in which enemy forces are located and fires into this area without feedback about the consequences of its fire.

(A5) Fire from surviving units is uniformly distributed over the area in which enemy forces are located, i.e. unaimed fire (in the sense of not being directed at specific enemy targets).

(A6) Each unit presents the same vulnerable area to enemy fire. This vulnerable area is much larger than the effective (or lethal) area of a single round of enemy fire, e.g. small arms fire at infantry targets. Additionally, the number of hits required for a kill obeys a geometric probability law.

NOTE: See text for explanation of notation.

the target (who suffers the attrition), while the second refers to the firer. We will always use this convention when there are double subscripts referring to both firer and target. Also, ν_Y denotes a single Y combatant's firing rate when he is engaging an acquired target, and P_{SSK} denotes a single-shot kill probability. Their product is then the rate at which acquired targets are killed by a single firer in this model. In Table 2.XIX, a_{V_X} denotes the vulnerable presented area of a single X combatant, A_X denotes the presented area occupied by the X force (and into which the Y force is assumed to fire), and $P(K|H)$ denotes the probability that a target is killed given that it is hit (i.e. conditional kill probability).

In Table 2.XX we summarize results for LANCHESTER's two classic models so that we can contrast their properties with each other. The term "aimed" fire is used in this table with the understanding that the target forces are "easily acquired," while the term "area" fire is used with the understanding that the target forces maintain a constant-area defense (cf. Table 2.XVII). Table 2.XX tells us that we may think of equations (2.2.1), i.e. LANCHESTER's equations for modern warfare, as arising when we fire only at live targets, while (2.4.1) arise when we fire at the original target positions with no feedback as to the consequences of our fire (see SCHREIBER [73]). Consequently, equations (2.4.1) implicitly involve "over-kill" (in the sense that one may continue to fire at dead targets), while equations (2.2.1) do not. Hence, we are not surprised that there is no advantage to the victor from concentrating forces in combat modelled by LANCHESTER's equations for area fire, but that there is for combat modelled by LANCHESTER's equations for modern warfare. Other results are similarly summarized. In the table, $\bar{x} = (x_0 + x_f)/2$ denotes X's "average" force level in the engagement, $x_c = x_0 - x_f$ denotes X's casualties in the engagement, and $u = x/y$ denotes the force ratio.

Table 2.XX. Comparison of LANCHESTER's Two Basic Combat Models.

	LANCHESTER's Equations for Modern Warfare	LANCHESTER's Equations for Area Fire
1. Simple Statement of Basic Model Assumption	"Aimed" Fire	"Area" Fire
2. Feedback Mechanism	Fire at Only Live Targets	Fire at Original Targets with No Feedback
3. Overkill?	NO	YES
4. State Equation	$b(x_0^2-x^2) = a(y_0^2-y^2)$	$b(x_0-x) = a(y_0-y)$
5. Concentration of Forces Advantageous for Victor?	YES	NO
6. Instantaneous Casualty-Exchange Ratio, $\frac{dx}{dy}$	$\frac{a}{b(x/y)}$	$\frac{a}{b}$
7. Overall Casualty-Exchange Ratio, $\frac{x_c}{y_c}$	$\frac{a}{b(\bar{x}/\bar{y})}$	$\frac{a}{b}$
8. Victory Predicted for Y When $\frac{x_0}{y_0}$ <	$\sqrt{\frac{a}{b}\left\{\frac{1-(f_{BP}^Y)^2}{1-(f_{BP}^X)^2}\right\}}$	$\frac{a}{b}\left\{\frac{1-f_{BP}^Y}{1-f_{BP}^X}\right\}$
9. Rate of Change of Force Ratio, $\frac{du}{dt}$	$b\{u^2-\frac{a}{b}\}$	$bx\{u-\frac{a}{b}\}$
10. Negative Rate of Change for Force Ratio When $\frac{x_0}{y_0}$ <	$\sqrt{\frac{a}{b}}$	$\frac{a}{b}$

In summary, Table 2.XX lists various results for and properties of LANCHESTER's two classic combat models. We should take these two models as limiting cases for the mutual attrition of two homogeneous forces: the modern-warfare equations (2.2.1) represent in some sense the "most effective" application of firepower (i.e. perfect feedback as to the consequences of one's fire), while the area-fire equations (2.4.1) represent a "less effective" one, with no feedback as to the consequences of one's fire. In other words, we may take equations (2.2.1) to represent the case in which fire is concentrated on individual targets, while equations (2.4.1) represent the case in which it is not.[37] Moreover, TAYLOR [84] has shown that these two types of target attrition processes yield quite different structures for optimal time-sequential fire-distribution policies in a more general model for combat against heterogeneous forces. We have already seen that these two attrition processes yield quite different returns to a commander from concentrating his forces.

Thus, these two basic models may be considered in some sense to be limiting cases for possible force-attrition processes. One is tempted to conjecture that they bound most real-world attrition processes, i.e. in some sense most real-world attrition processes lie between these two extreme points. Furthermore, they form the basis for essentially all further developments in the LANCHESTER theory of combat and yield important insights into the behavior of more complex models.[38]

2.12. A Classification Scheme for Homogeneous-Force LANCHESTER-Type Attrition Processes and Some Additional Functional Forms for Attrition Rates.

As we have seen above for LANCHESTER's two basic combat models, several different sets of physical assumptions may be hypothesized to yield the same functional form for an attrition rate. Consequently, it is more convenient to refer to a model for combat between two homogeneous forces in terms of the functional forms for the two attrition rates than to refer in terms of the assumptions (as we have done above). Let us now introduce a very convenient shorthand for referring to such homogeneous-force LANCHESTER-type combat models. It basically involves using a two-part descriptor $X|Y$, where X describes the attrition rate for the X force and similarly for Y. X and Y take on their values according to the type of proportionality for the various terms in a side's attrition rate. This proportionality is expressed in terms of the number of firers (denoted as F) and/or the number of targets (denoted as T). If the attrition rate is independent of the numbers of firers and targets, we use the letter C (for constant attrition rate). When there is more than one term in a side's attriton rate, the same approach is applied to each term, with a plus sign separating each component term of the attrition rate.

Let us now consider some examples to illustrate this shorthand. For example, for LANCHESTER's equations of modern warfare (2.2.1), the X-force attrition rate is $(-dx/dt) = ay$ so that it is proportional to only the number of enemy firers (and similarly for the Y-force attrition rate). Consequently, we will refer to it as a $F|F$ LANCHESTER-type attrition process (or, simply, $F|F$ attrition). Similarly, LANCHESTER's equations for area fire (2.4.1) represent $FT|FT$ attrition, since each side's attrition rate is proportional to the product of the number of firers and the number of targets. As a final example (with two terms in each side's attrition rate),

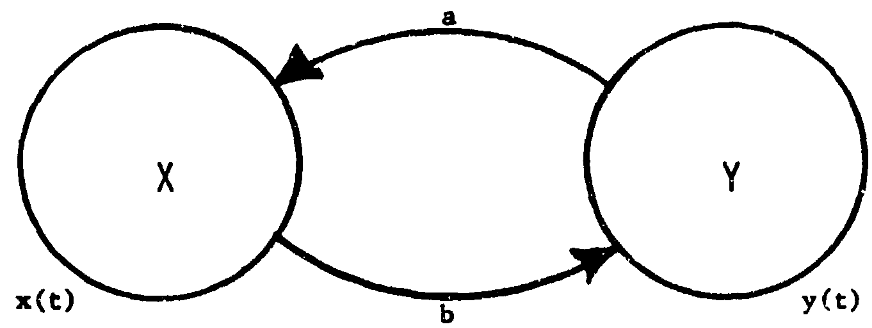
the equations

$$\begin{cases} \dfrac{dx}{dt} = -\,ax^{\mu_1}y^{\nu_1} - \beta x^{\mu_2}y^{\nu_2}\,, \\ \\ \dfrac{dy}{dt} = -\,bx^{\mu_3}y^{\nu_3} - \alpha x^{\mu_2}y^{\nu_4}\,, \end{cases}$$

will be said to represent $\left(F^{\nu_1}T^{\mu_1} + F^{\nu_2}T^{\mu_2}\right)\Big|\left(F^{\mu_3}T^{\nu_3} + F^{\mu_4}T^{\nu_4}\right)$ LANCHESTER-type attrition.

Figure 2.14 shows various attrition-rate functional forms that have been considered in the literature of the LANCHESTER theory of combat. We have used the above shorthand notation for referring to these various attrition processes in the figure. Also shown for each process are the state equation (if not too complicated) and the first person (known to this author) to have considered it. Table 2.XXI gives an enumeration of authors who have studied each of these various "basic" attrition-rate processes.

Let us now briefly examine the various sets of physical assumptions that have been hypothesized to yield the five basic attrition-rate functional forms shown in Figure 2.14. Conditions hypothesized to yield the $F|F$ and $FT|FT$ attrition processes have been discussed previously in Section 2.11 (see, for example, Table 2.XVII), and conditions for the $F|FT$ process (equivalently, the $FT|F$ process), of course, are just a combination of these two sets, with one set applying for each side. For example, BRACKNEY [15] has hypothesized that the $F|FT$ attrition process occurs for an assault by the X forces on defensive Y positions, in which the defenders use aimed fire (with X targets readily acquired by virtue of their "assault" posture) and so do the attackers, only their search time for Y targets is relatively large (and inversely proportional to enemy troop density) by virtue of the enemy's remaining under cover in their defensive positions. In other words, assumptions ($\tilde{A}1$) and ($\tilde{A}2$) of Table 2.XVII apply to X, while (M1) and (M2) apply to Y.



ATTRITION PROCESS	DIFFERENTIAL EQUATIONS	STATE EQUATION
F|F	$\frac{dx}{dt} = -ay$ $\frac{dy}{dt} = -bx$	LANCHESTER (1914) $b(x_0^2-x^2) = a(y_0^2-y^2)$ square law
FT|FT	$\frac{dx}{dt} = -axy$ $\frac{dy}{dt} = -bxy$	LANCHESTER (1914) $b(x_0-x) = a(y_0-y)$ linear law
F|FT	$\frac{dx}{dt} = -ay$ $\frac{dy}{dt} = -bxy$	BRACKNEY (1959) $\frac{b}{2}(x_0^2-x^2) = a(y_0-y)$ mixed law
T|T	$\frac{dx}{dt} = -ax$ $\frac{dy}{dt} = -by$	PETERSON (1953) $b \ln \frac{x_0}{x} = a \ln \frac{y_0}{y}$ logarithmic law
(F+T)|(F+T)	$\frac{dx}{dt} = -ay - \beta x$ $\frac{dy}{dt} = -bx - \alpha y$	MORSE and KIMBALL (1951) (generally very complicated)

Figure 2.14. Various functional forms for attrition rates that have been considered in the LANCHESTER-combat-theory literature.

TABLE 2.XXI. Authors Who Have Studied Various Basic Attrition-Rate Processes.

Attrition Process	
F|F	LANCHESTER (1914) KOOPMAN (1940's; <u>see</u> MORSE and KIMBALL (1951))
FT|FT	LANCHESTER (1914) MARADUDIN and G. WEISS (1958) G. WEISS (1963)
F|FT	BRACKNEY (1959) DEITCHMAN (1962)
T|T	PETERSON (1953, 1967) CLARK (1969) HELMBOLD (1965) H. K. WEISS (1966)
(F+T)|(F+T)	MORSE and KIMBALL (1951) ISBELL and MARLOW (1956A) BACH, DOLANSKY, and STUBBS (1962) TAYLOR and PARRY (1975)

Also, DEITCHMAN [22] has used the $F|FT$ attrition model for insurgency operations (i.e. guerrilla warfare) to represent the ambush of X-force counterinsurgents by Y-force guerrillas. He hypothesized that (M1) and (M2) hold for the Y force, which fires on the X force, "caught in the open," but that the ambushed X force can only return area fire, since its members do not know the exact positions of individual Y ambushers and consequently return fire into only the general area known to be occupied by the enemy.

PETERSON [69;70] has hypothesized that $T|T$ attrition, i.e.

$$\frac{dx}{dt} = -ax, \qquad \text{and} \qquad \frac{dy}{dt} = -by, \tag{2.21.1}$$

characterizes the early stages of a small-unit engagement in which the vulnerability of a force dominates its ability to acquire enemy targets. In other words, $T|T$ attrition occurs when the exposure of individual weapons to be acquired as targets determines the occurrences of initial casualties.

PETERSON [69] introduced this model to extend the available choice of basic combat models and also because it does fit limited data for a certain type of engagement, i.e. a tactical situation in which all weapons of the two forces are within effective range of the enemy but when (due to cover, concealment, or expert camouflage) no two opposing weapons are actually intervisible. In such a situation, it is not unreasonable to assume that the probability that the first unit to betray his cover, concealment, or camouflage is in the X force is given by the ratio $ax/(ax + by)$, whence follows (2.12.1) (see Chapter 4). However, once the battle actually begins, this model is no longer applicable.

WEISS [102] has suggested that force vulnerability may become the dominant factor in causing losses as combat units increase in size and become increasingly inefficient. G. CLARK [20] has used this $T|T$ attrition model (2.12.1) for the early stages of a small-unit engagement in his COMAN model.

The last attrition-rate functional form shown in Table 2.14 is that of $(F + T)|(F + T)$ attrition, i.e.

$$\frac{dx}{dt} = -ay - \beta x, \qquad \text{and} \qquad \frac{dy}{dt} = -bx - \alpha y. \qquad (2.12.2)$$

Two situations that have been hypothesized to yield the above equations are (see Figure 2.15):

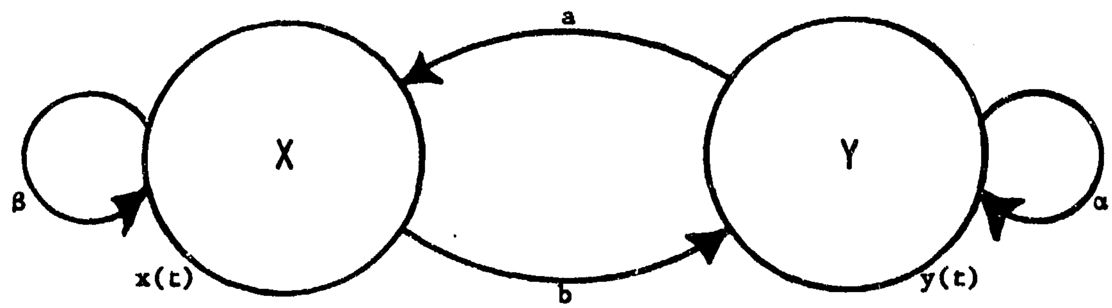
(S1) $F|F$ attrition in combat between two homogeneous forces with "operational" losses [3;64],

(S2) $F|F$ attrition in combat between two homogeneous primary forces (see WEISS [100]) with superimposed effects of supporting fires not subject to attrition [95].

In the first situation (S1), for example, the term (βx) in X's loss rate, i.e. $(-dx/dt)$, represents "operational" losses, i.e. losses due to causes other than enemy action [3] (e.g. losses due to sickness, accidents, desertions, etc.).[39] In other words, the model holds that a force suffers a certain amount of casualties due to its very size. In the second situation (S2), it is assumed that $F|F$ attrition holds between the primary fighting forces, e.g. infantries, and that the supporting weapons employ area fire against enemy infantry (again see Figure 2.15).

Let us note that the state equation is quite simple (and is trivially derived) for each of the first four attrition processes shown in Figure 2.14.

(a) operational losses



(b) combat with supporting fires not subject to attrition

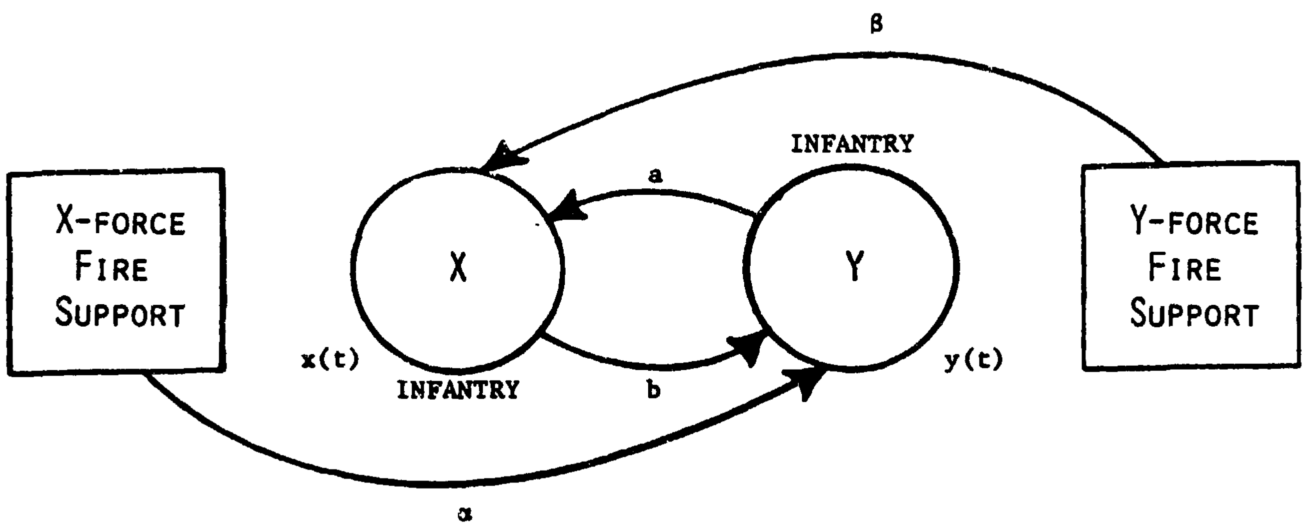


Figure 2.15. Two different combat situations that ha‧ - been hypothesized to yield (F+T)|(F+T) attrition.

However, the state equation for the last one, the $(F + T)|(F + T)$ attrition process, is generally quite complicated, namely [95]

$$y(t)\{\theta + (\frac{\beta-\alpha}{2})\} - bx(t) =$$

$$[y_0\{\theta + (\frac{\beta-\alpha}{2})\} - bx_0] \left[\frac{y_0\{\theta - (\frac{\beta-\alpha}{2})\} + bx_0}{y(t)\{\theta - (\frac{\beta-\alpha}{2})\} + bx(t)} \right]^{\nu} , \qquad (2.12.3)$$

where $\theta = \sqrt{ab + [(\beta-\alpha)/2]^2}$ and $\nu = \{\theta - (\alpha+\beta)/2\}/\{\theta + (\alpha+\beta)/2\}$. However, as first noted by Taylor and Parry [95], when $ab = \alpha\beta$, then $\theta = (\alpha+\beta)/2$ and $\nu = 0$, so that (2.12.3) becomes

$$\beta y(t) - bx(t) = \beta y_0 - bx_0 \qquad \text{for } ab = \alpha\beta , \qquad (2.12.4)$$

which is a totally unexpected result. Later in this book we will give some insights as to why this complicated state equation (2.12.3) for the $(F + T)|(F + T)$ model (2.12.2) reduces to the "linear law" (2.12.4) in this special case.

A general form for homogeneous-force attrition rates (which yields the square, linear, and logarithmic laws as special cases) has been given by HELMBOLD [36], who hypothesized that the larger force suffers inefficiences of scale when force sizes are grossly unequal.[40] He has emphasized that LANCHESTER's classic equations for modern warfare (2.2.1), i.e. the $F|F$ attrition model, imply that no matter how unequal the opposing strengths may be, the full destructive capability of each side can be focused with undiminished effects on the enemy. However, sheer limitations of available

space, to say nothing of terrain-masking and reaction-time effects, may well prevent the larger force from using its full destructive capability.

In consonance with the above line of reasoning, HELMBOLD [36] has suggested the following LANCHESTER-type equations

$$\frac{dx}{dt} = -a \cdot g(\frac{x}{y}) \cdot y \ , \qquad \text{and} \qquad \frac{dy}{dt} = -b \cdot h(\frac{y}{x}) \cdot x, \tag{2.12.5}$$

where, for example, $g(x/y)$ is a function that is used to modify the fire effectiveness of an individual Y combatant at extreme force ratios and similarly for $h(y/x)$. HELMBOLD argued that the effectivenss-modification functions should satisfy the following three requirements

(R1) $g(1) = h(1) = 1$ ((2.12.5) reduces to (2.2.1) for forces of equal size),

(R2) $g(q) = h(q)$ (same inefficiencies of scale for each side),

(R3) $g(q)$ is a strictly increasing function of its argument.

Hence, (2.12.5) becomes

$$\frac{dx}{dt} = -a \cdot h(\frac{x}{y}) \cdot y, \qquad \text{and} \qquad \frac{dy}{dt} = -b \cdot h(\frac{y}{x}) \cdot x \ , \tag{2.12.6}$$

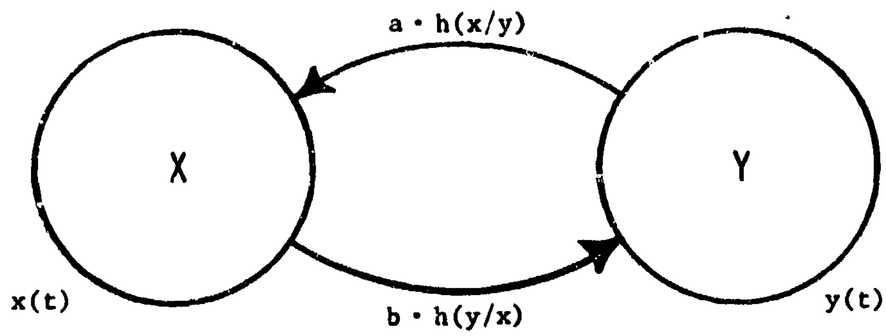
which we will refer to as the equations for generalized HELMBOLD-type combat (see Figure 2.16). Here, the effectiveness-modification function $h(z)$ has the following properties:

(P1) $h(z)$ is a strictly increasing function of its argument,

(P2) $h(1) = 1$.

HELMBOLD [36] also considered the special case of (2.12.6) in which $h(z)$ is a power function of its argument,[41] i.e. $h(z) = u^c$. Then, (2.12.6) becomes

HELMBOLD (1965)



$$\frac{dx}{dt} = -a\,h\left(\frac{x}{y}\right) y, \qquad \frac{dy}{dt} = -b\,h\left(\frac{y}{x}\right) x$$

Figure 2.16. Generalized Helmbold-type combat which incorporates inefficiencies of scale for the larger force when force sizes are grossly unequal.

$$\frac{dx}{dt} = -a\cdot\left(\frac{x}{y}\right)^{c}\cdot y \ , \qquad \text{and} \qquad \frac{dy}{dt} = -b\cdot\left(\frac{y}{x}\right)^{c}\cdot x \ , \tag{2.12.7}$$

which we will refer to as the equations for HELMBOLD-type combat[42] (see Figure 2.17). It follows that the instantaneous exchange ratio, dx/dy, is given by[43]

$$\frac{dx}{dy} = \frac{a}{b}\left(\frac{x}{y}\right)^{2c-1} = \frac{a}{b}\left(\frac{y}{x}\right)^{d-1} \ , \tag{2.12.8}$$

where $d = 2(1-c)$. Hence, the state equation may be written (for $d \neq 0$) as

$$b(x_0^d - x^d) = a(y_0^d - y^d) \ , \tag{2.12.9}$$

and for $d = 0$

$$b \ \ln \frac{x_0}{x} = a \ \ln \frac{y_0}{y} \ . \tag{2.12.10}$$

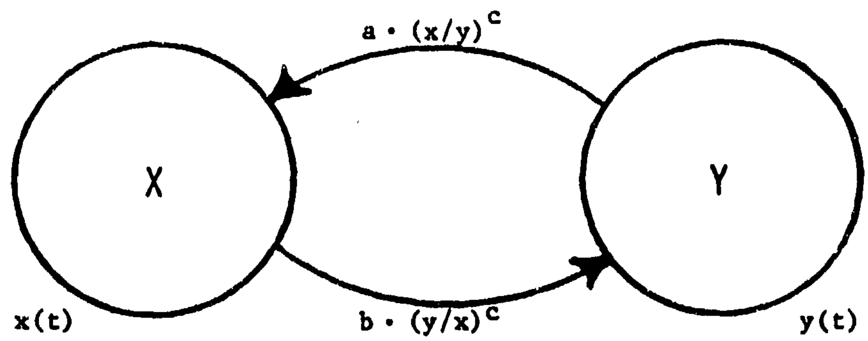
Thus, the equation for HELMBOLD-type combat yield the square law when $c = 0$, the linear law when $c = 1/2$, and the logarithmic law when $c = 1$ (see Figure 2.17).

Moreover, there is an intimate relationship between the equations for HELMBOLD-type combat (2.12.7) and those of LANCHESTER for modern warfare (2.2.1). It is convenient, however, to first introduce the "Weiss parameter" W defined by

$$W = d/2 = 1 - c \ , \tag{2.12.11}$$

and to write (2.12.7) as

$$\frac{dx}{dt} = -a\cdot\left(\frac{x}{y}\right)^{1-W}\cdot y \ , \qquad \text{and} \qquad \frac{dy}{dt} = -b\cdot\left(\frac{y}{x}\right)^{1-W}\cdot x \ , \tag{2.12.12}$$



Model: $\frac{dx}{dt} = -a\left(\frac{x}{y}\right)^c y, \qquad \frac{dy}{dt} = -b\left(\frac{y}{x}\right)^c x$

Instantaneous Exchange Ratio: $\frac{dx}{dy} = \frac{a}{b}\left(\frac{x}{y}\right)^{2c-1} = \frac{a}{b}\left(\frac{y}{x}\right)^{d-1}$

State Equation: $b(x_0^d - x^d) = a(y_0^d - y^d)$

SPECIAL CASES (WEISS, 1966B)

EXPONENTS		LAW	COMMENTS
c	d		
0	2	SQUARE	CONCENTRATE
1/2	1	LINEAR	
3/4	1/2	SQUARE ROOT	EXERCISE ECONOMY OF FORCE
1	0	LOGARITHMIC	"ALL FORCES ARE EQUAL"

Figure 2.17. HELMBOLD-type combat which incorporates inefficiencies of scale for the larger force when force sizes are grossly unequal.

where $W \in (0,1]$ for $c \in [0,1)$. Introducing the force ratio $u = x/y$, we obtain the <u>force-ratio equation</u>

$$\frac{du}{dt} = u^{1-W}(bu^{2W} - a) \quad . \tag{2.12.13}$$

The form of the equation (2.12.13) suggests letting $v = u^W$. Doing this, we may transform the force-ratio equation into the following RICCATI equation

$$\frac{dv}{dt} = W(bv^2 - a) \quad , \tag{2.12.14}$$

with initial condition $v(0) = x_0^W/y_0^W$. Since we have encountered a RICCATI equation for $v = x^W/y^W$, we know that both x^W and y^W satisfy linear differential equations (<u>see</u> Appendix A.3). Setting $p = x^W$ and $q = y^W$, we find that

$$\begin{cases} \dfrac{dp}{dt} = -Waq & \text{with} \quad p(0) = x_0^W \; , \\ \\ \dfrac{dq}{dt} = -Wbp & \text{with} \quad q(0) = y_0^W \; . \end{cases} \tag{2.12.15}$$

The result (2.12.15) is highly significant, since it shows that the nonlinear differential-equation model of HELMBOLD-type combat (2.12.7) can be transformed into the familiar linear model (2.2.1) so that all the known results for the linear model can be invoked. In particular, it follows that (2.12.9) holds (since $p = x^{d/2}$ and $q = y^{d/2}$) and

$$x^W(t) = x_0^W \cosh W\sqrt{ab}\, t - y_0^W \sqrt{\frac{a}{b}} \sinh W\sqrt{ab}\, t . \qquad (2.12.16)$$

Thus, for the model of HELMBOLD-type combat, one can readily answer questions (Q1) through (Q7) posed in Section 2.2 above. For example, Y will win a fixed-force-level-breakpoint battle in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt[d]{\frac{a}{b}\frac{\{1 - (f_{BP}^Y)^d\}}{\{1 - (f_{BP}^X)^d\}}} . \qquad (2.12.17)$$

As we pointed out in Section 2.9, many different differential-equation combat models can yield LANCHESTER's linear law (2.4.3) (including the $(F + T)|(F + T)$ model (2.12.2) when $ab = \alpha\beta$). We did not call (2.4.1) the equations for a linear-law attrition process for this reason. When $c = 1/2$ and consequently $d = 1$, (2.12.9) becomes the linear law, but the X force level as a function of time is given (implicitly) by

$$\sqrt{x(t)} = \sqrt{x_0} \cosh(\sqrt{ab}\, t/2) - \sqrt{y_0\, a/b} \sinh(\sqrt{ab}\, t/2) , \qquad (2.12.18)$$

which should be contrasted with the corresponding result (2.4.7) for the FT|FT attrition process. In particular, it should be noted that (2.12.18) implies that, for example, the X force can be annihilated in finite time, whereas this outcome is impossible for "linear-law" combat modelled with (2.4.1) (see Proposition 2.4.2).

Let us finally note that the above transformation of the non-linear equations for HELMBOLD-type combat (2.12.12) (equivalently, (2.12.7)) into a linear differential-equation model also holds for time-dependent

attrition-rate coefficients. Moreover, (2.12.7) is the only such non-linear combat model with a "separable" efficiency factor (i.e. $h(x/y) = f(x)/g(y)$ in (2.12.6)) that can be transformed into the $F|F$ attrition model (*see* Section 6.11 below).

PROBLEMS for Chapter 2

1. What did F. W. LANCHESTER hope to prove with his simple mathematical models of combat?

2. What are three important characteristics of a good analytical model? (Short answer in words is all that is sought. You may want to refer back to Chapter 1.)

3. With reference to LANCHESTER's original work, what is the major difference between the conditions under which the $FT|FT$ attrition process has been hypothesized to occur and those for the $F|F$ attrition process? (A single phrase for each will suffice here.)

4. Fill in the missing entries in the below table that illustrates how under "modern conditions" of warfare there is an advantage from concentrating forces. For these computations assume:

$$\frac{a}{b} = E = 0.25, \quad x_0 = 100, \quad \text{and} \quad x_f = 0,$$

where x_0 denotes the initial value for the X force level and x_f denotes its final value.

y_0	200	250	300	400	500	1000
y_f	0	---	223.6	---	---	----
Y's loss	200	---	-----	---	41.7	----

5. Redo the table that you constructed for Problem 4, but instead of a fight to the finish, consider a fixed-force-level-breakpoint battle with $f^X_{BP} = f^Y_{BP} = 0.25$, where $x_{BP} = f^X_{BP}\, x_0$ and similarly for y_{BP}. Thus, your input data will be $a/b = E = 0.25$, $x_0 = 100$, $x_f = x_{BP} = 25$, and $f^X_{BP} = f^Y_{BP} = 0.25$, with the table containing entries for $y_0 = 200$, 250, 300, 400, 500, and 1000.

6. Consider combat between two homogeneous forces modelled by the following F|F LANCHESTER-type equations (for x and $y > 0$)

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with} \quad x(0) = x_0, \\[2ex] \dfrac{dy}{dt} = -bx & \text{with} \quad y(0) = y_0, \end{cases}$$

where a and b denote positive constants.

<u>Part a</u>. What assumptions have been hypothesized to yield the above combat dynamics? (Only one set of simple assumptions sought.)

Part b. What are the constants a and b called in the above LANCHESTER-type combat model?

Part c. What are the dimensions of a?

Part d. What is the quantity $\sqrt{ab}$ called? The quantity a/b?

Part e. What is the X force level given by?

Part f. Let Y attack the X force, which defends. How are X's fractional casualties per unit time related to the force ratio of the attacker to the defender? Sketch a plot of this relationship. How is the constant a related to this plot? [HINT: Observe that X's fractional casualties per unit time are given by $(-1/x)\, dx/dt$.]

7. Let us further consider the LANCHESTER-type combat model of Problem 6.

Part a. If $a = 0.06$ X casualties/minute/Y combatant, $b = 0.01$ Y casualties/minute/X combatant, $x_0 = 200$, and $y_0 = 100$, who will win a fight to the finish?

Part b. For the data given in Part a above, how long will it take for the loser to be annihilated?

Part c. For the data given in Part a above, plot the X force level $x(t)$ as a function of time. What is $x(t)$ for $t = 60$ minutes?

Part d. For the data given in Part a above, plot the Y force level $y(t)$ as a function of time. What is $y(t)$ for $t = 60$ minutes?

Part e. If a reserve force of 70 X combatants (assume that these reinforcements are identical to the original members of the X force) arrives after 30 minutes and is immediately committed to battle, who will win this fight to the finish? What would have been the outcome if X could have initially committed his reserve?

Part f. Who will win a fight to the finish if $a = 0.09$ X casualties/minute/Y combatant, $b = 0.02$ Y casualties/minute/X combatant, $x_0 = 300$, and $y_0 = 100$? What is x when $y = 75$? When $y = 50$? When $y = 25$? When $y = 0$?

Part g. If $a = 0.01$ X casualties/minute/Y combatant, $b = 0.01$ Y casualties/minute/X combatant, $x_0 = 300$, and $y_0 = 100$, who will win a fight to the finish? Who will win if $x_0 = 350$? If $x_0 = 400$? If $x_0 = 500$?

8. Let us further consider the LANCHESTER-type combat modal of Problems 6 and 7, only this time we will assume that the engagement is a fixed-force-level breakpoint battle. As usual, we will represent the force-level breakpoints as $x_{BP} = f_{BP}^{X} x_0$ and $y_{BP} = f_{BP}^{Y} y_0$.

Part a. If $a = 0.01$ X casulaties/minute/Y combatant, $b = 0.04$ Y casualties/minute/X combatant, $x_0 = 100$, $Y_0 = 225$, $f_{BP}^{X} = 0.5$, and $f_{BP}^{Y} = 0.7$, who will win a fixed-force-level breakpoint battle?

Part b. For the data given in Part a above, how long will it take for the loser to read his breakpoint?

Part c. For the data given in Part a above, plot the X force level $x(t)$ as a function of time. What is $x(t)$ for $t = 45$ minutes?

Part d. For the data given in Part a above, plot the Y force level $y(t)$ as a function of time. What is $y(t)$ for $t = 45$ minutes?

Part e. Who will win a fixed-force-level-breakpoint battle if $a = 0.01$ X casualties/minute/Y combatant, $b = 0.05$ Y casualties/minute/X combatant, $x_0 = 100$, $y_0 = 300$, $f_{BP}^{X} = 0.5$, and $f_{BP}^{Y} = 0.7$? Who will win if $y_0 = 250$?

Part f. Who will win a fixed-force-level-breakpoint battle if $a = 0.001$ X casualties/minute/Y combatant, $b = 0.01$ Y casualties/minute/X combatant, $x_0 = 100$, $y_0 = 400$, $f_{BP}^{X} = 0.4$, and $f_{BP}^{Y} = 0.65$? Who will win if $y_0 = 350$?

Part g. If $a = 0.06$ X casualties/minute/Y combatant, $b = 0.01$ Y casualties/minute/X combatant, $f_{BP}^{X} = 0.65$, and $f_{BP}^{Y} = 0.5$, what initial force ratio is required for X to win a fixed-force-level-breakpoint battle? What do these numbers suggest to you as far as who is the attacker and who is the defender? If you were the commander of the X force, what initial force ratio would you want before you engaged the enemy? Why?

9. Now let both sides receive replacements continuously over time at constant rates. The above combat model then becomes

$$\begin{cases} \dfrac{dx}{dt} = -ay + r & \text{with } x(0) = x_0, \\ \dfrac{dy}{dt} = -bx + s & \text{with } x(0) = y_0, \end{cases}$$

where the positive constants r and s denote the replacement rates for the X and Y forces, respectively. What is the state equation for the above LANCHESTER-type combat model with continuous replacements?

10. The model of the previous problem possesses the conceptual shortcoming that both sides have essentially been assumed to possess unlimited reserves. How would you modify the model of Problem 9 to reflect the situation in which both sides have available only limited pools of manpower out of which to draw replacements? Let R_0 denote the total number of replacements that X can commit to battle, and similarly let S_0 denote the total number of replacements available to Y.

11. S. J. DEITCHMAN [22] has proposed the following LANCHESTER-type model to represent the ambush of X-force counterinsurgents by Y-force guerrillas in guerrilla-warfare operations

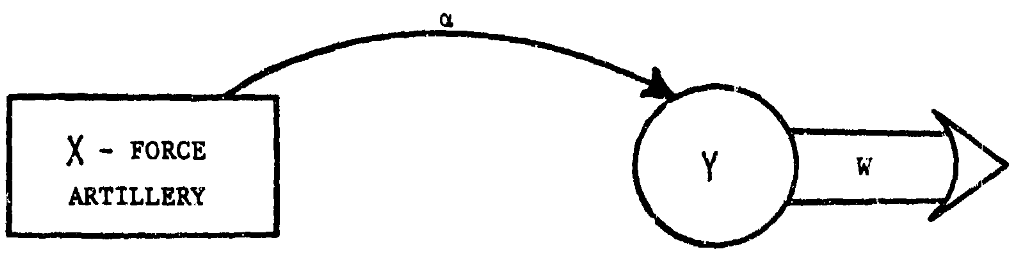
$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = x_0, \\ \dfrac{dy}{dt} = -bxy & \text{with } y(0) = y_0, \end{cases}$$

where a and b denote LANCHESTER attrition-rate coefficients that are positive constants. He hypothesized that the ambushers (i.e. the Y force) would use aimed fire from well-chosen and concealed positions, and that the ambushees (i.e. the X force) would only be able to return area fire into the general region occupied by the enemy because they (i.e. the ambushees) have been "caught in the open" and do not know the positions of individual Y ambushers.

Part a. What is the state equation for DEITCHMAN's ambush model given by?

Part b. What condition on the initial force levels predicts victory for the abmusher in a fight to the finish?

12. Consider combat between homogeneous X and Y forces



in which the artillery of the X force delivers area fire against the Y force, which occupies a constant area. This artillery is out of firing range of the Y force and hence suffers no attrition. Consequently, the LANCHESTER-type equation that describes this combat-attrition process is

$$\frac{dy}{dt} = -\alpha(t)y \qquad \text{with} \quad y(0) = y_0,$$

where $\alpha(t)$ denotes a time-dependent LANCHESTER attrition-rate coefficient.

Part a. What is the Y force level $y(t)$ given by?

Now let the fire effectiveness of the X-force artillery be constant (i.e. let α = constant) and let the Y force withdraw from their original positions at a variable rate, denoted as $W(t)$, to new positions that are free from the effects of the enemy's artillery fire. The corresponding LANCHESTER-type combat equation then becomes (for $y > 0$)

$$\frac{dy}{dt} = -\alpha y - W(t) \qquad \text{with} \quad y(0) = y_0 ,$$

where $W(t) > 0$.

Part b. What is the Y force level $y(t)$ now given by?

Part c. Denote the number of casualties of the Y force as $c(t)$. What is $c(t)$ given by?

Now let the withdrawal rate of the Y force be constant so that the LANCHESTER-type combat model becomes

$$\frac{dy}{dt} = \begin{cases} -\alpha y - W & \text{for } y > 0, \\ 0 & \text{for } y = 0, \end{cases} \qquad \text{with } y(0) = y_0,$$

where $W > 0$.

<u>Part d</u>. What is the Y force level $y(t)$ now given by?

<u>Part e</u>. If $\alpha = 0.1$ Y casualties/minute/Y combatant, $W = 10$ men/minute, and $y_0 = 100$; will an air strike after 7.5 minutes help the X force?

<u>Part f</u>. If $\alpha = 0.1$ Y casualties/minute/Y combatant, $W = 10$ men/minute, and $y_0 = 100$; how many casulaties will the Y force suffer?

13. To each of the entries on the left below, match the entry on the right to which it is most closely related. Do this by placing the latter of the appropriate entry on the right in the answer space on the left.

(1) ____ $\frac{dx}{dy}$,

(2) ____ $\begin{cases} \frac{dx}{dt} = -ay, \\ \frac{dy}{dt} = -bxy, \end{cases}$

(3) ____ $\begin{cases} \frac{dx}{dt} = -axy, \\ \frac{dy}{dt} = -bx, \end{cases}$

(4) ____ $\begin{cases} \frac{dx}{dt} = -ay - \beta x, \\ \frac{dy}{dt} = -bx - \alpha y, \end{cases}$

(5) ____ $\sqrt{ab}$

(6) ____ $\frac{a}{b(\bar{x}/\bar{y})}$

(7) ____ $\frac{du}{dt} = bu^2 - a$,

(8) ____ $\left(-\frac{dx}{dt}\right)$,

(9) ____ $\frac{dx}{dt} = -a\left(\frac{x}{y}\right)^c y$,

(10) ____ $\left(-\frac{1}{x}\frac{dx}{dt}\right)$.

a. T|T attrition process,

b. FT|T attrition process,

c. FT|F attrition process,

d. "aimed-fire" combat with supporting fires not subject to attrition,

e. state equation,

f. force-ratio equation for F|F attrition process,

g. force-ratio equation for (F+T)|(F+T) attrition process,

h. force-ratio equation for FT|FT attrition process,

i. force-annihilation-prediction condition for F|F attrition process,

j. force-level change per unit time,

k. casualties per unit time,

l. fractional casualties per unit time,

m. overall casualties for X force,

n. total replacements,

o. LANCHESTER-type equations for a skirmish,

p. instantaneous casualty-exchange ratio,

q. unit deterioration due to attrition,

r. Y force ambushing the X force,

s. inefficiencies of scale for larger force when force sizes are grossly unequal,

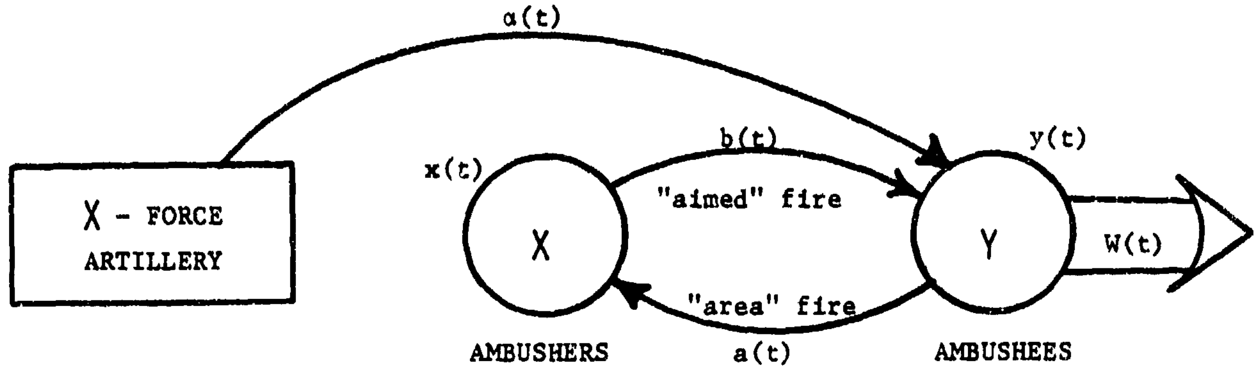
t. overall casualty-exchange ratio for F|F attrition process,

u. overall casualty-exchange ratio for FT|FT attrition process,

v. relative fire effectiveness,

w. intensity of combat.

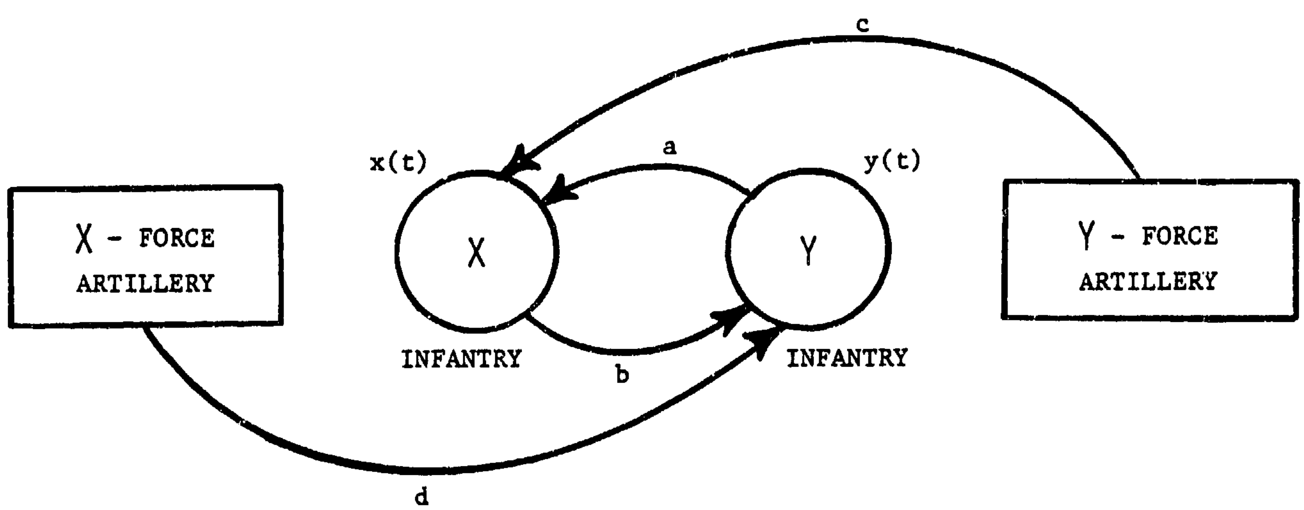
14. Consider the ambush of a homogeneous Y force by a



homogeneous X force, both of which are armed with small arms. The X force uses aimed fire, with an associated time-dependent LANCHESTER attrition-rate coefficient denoted as $b(t)$; and the Y force returns area fire, with an associated time-dependent LANCHESTER attrition-rate coefficient denoted as $a(t)$. In other words, the X force ambushes with aimed fire, the Y force returns area fire, and on each side the fire effectiveness of an individual firer changes over time during the fire fight. Moreover, the X force has called for supporting fire from artillery that is out of range of any return fire from the Y force and that consequently suffers no attrition. This artillery causes attrition to the Y force at a rate proportional to the Y force level with an associated "constant" of proportionality $\alpha(t)$. This attrition-rate coefficient is time dependent and accounts for the number of firing tubes (i.e. artillery pieces). Because of the ambush and also this fire support, the Y force wants to terminate the engagement, and consequently it gradually disengages from combat with the X force (including its fire

support). Let $W(t) > 0$ denote the time-dependent rate at which the Y force withdraws from this engagement to a position that is out of range of all enemy firers. Let $x(t)$ denote X's force level (with initial value denoted as x_0), and similarly let $y(t)$ denote Y's force level (with initial value denoted as y_0). Consider only that phase of the engagement during which both x and $y > 0$. What are appropriate LANCHESTER-type equations for the rates of change of the X and Y force levels?

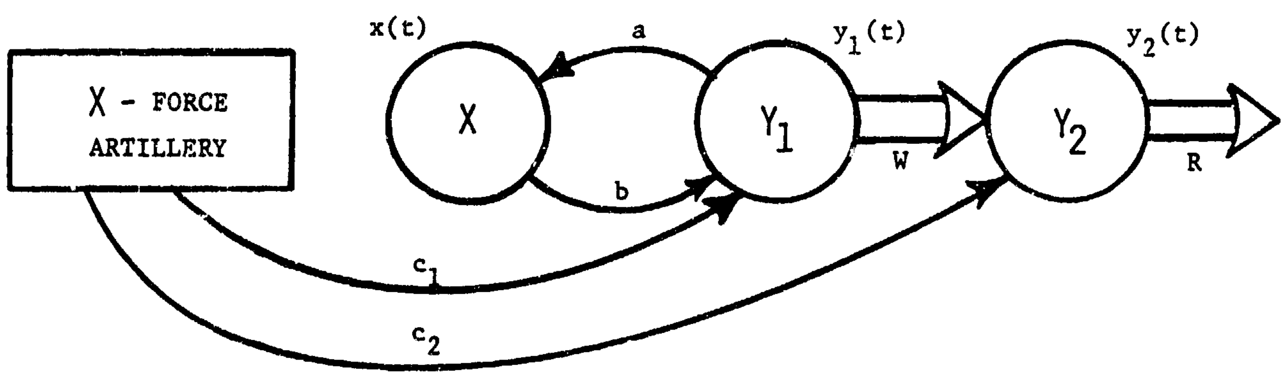
15. Consider LANCHESTER-type combat between homogeneous



X and Y infantry forces with supporting artillery not subject to attrition. Each member of the Y force uses aimed fire to destroy the X force at a rate a. Similarly, each member of the X force uses aimed fire to destroy the Y force at a rate b. Both sides have artillery, which does not suffer any attrition and delivers "area" fire against the enemy infantry. The Y-force artillery fires at a constant rate and

causes attrition to the enemy infantry at a rate proportional to the X force level with an associated constant of proportionality c (which accounts for the constant number of firing tubes). Similarly, the X-force artillery fires at a constant rate and causes attrition to the enemy infantry at a rate proportional to the Y force level with an associated constant of proportionality d (which accounts for the constant number of firing tubes). Let $x(t)$ denote the force level of X's infantry (with initial value denoted as x_0), and similarly let $y(t)$ denote the force level of Y's infantry (with initial value denoted as y_0). Consider only that phase of the engagement during which both x and $y > 0$. What are appropriate LANCHESTER-type equations for the rates of change of the X and Y force levels?

16. Consider LANCHESTER-type combat between an X force and a Y force



(initially all in bunkers). Denote the initial Y force level as y_0. Also, denote that part of the Y force which is in the fortified position (i.e. in the bunkers) as Y_1. Each member of the X force uses aimed

fire to destroy the Y_1 force at a rate denoted as b. Similarly, each member of the Y_1 force uses aimed fire to destroy the X force at a rate denoted as a. Additionally, the Y_1 force withdraws from the bunkers at a rate W to become withdrawing troops, denoted as Y_2. The Y_2 force does not exchange fire with the X force, but Y_2 is subject to receive supporting fire from X's artillery. Members of the Y_2 force retreat further to positions that are not vulnerable to the X-force artillery fire. Let the rate at which the vulnerable Y_2 force is diminished by this retreat by denoted as R (where $0 < R < W$). The artillery of the X force does not suffer any attrition and divides its area fire between Y_1 and Y_2. Firing at a constant rate, the artillery causes attrition to Y_1 at a rate proportional to the Y_1 force level with an associated constant of proportionality c_1 (which accounts for both the constant number of firing tubes and the allocation of fire) and similarly to Y_2 with an associated constant of proportionality c_2. Let $x(t)$ denote X's force level (with initial value denoted as x_0), $y_1(t)$ denote Y_1's force level, and $y_2(t)$ denote Y_2's force level. Consider only that phase of the engagement during which x, y_1, and $y_2 > 0$. What are appropriate LANCHESTER-type equations for the rates of change of the X, Y_1, and Y_2 force levels?

17. Consider a homogeneous X force that attacks in two echelons a homogeneous Y force in a hasty-defense position. Assume that the F|F LANCHESTER-type equations (2.2.1) describe the attrition process of the first echelon of the X force against the Y defenders in this attack. The two echelons of the X force move in such a way that the second echelon does not inflict

nor sustain any casualties while the first echelon is fighting, but that the second echelon can quickly replace the first at the appropriate time during the attack (assume that the time required to effect this replacement is negligible). Furthermore, assume that for this attack $a = 0.05$ X casualties/minute/Y combatant, $b = 0.01$ Y casualties/minute/X combatant, the initial strength of the first echelon of the X force is 2000, that of the second echelon of the X force is 1250, that the Y force will withdraw when it has suffered 75 percent casualties, and that the first echelon of the X force fights until it reaches 25 percent of its initial strength at which time it is replaced in toto by the second echelon, which fights on with the same combat effectiveness (and vulnerability) per man and also the same engagement-termination conditions as the first echelon. Plot the X and Y force levels $x(t)$ and $y(t)$ as a function of time for this two-echelon attack of X against Y.

18. COL. T. S. SCHREIBER [73] has proposed the following simple LANCHESTER-type model in order to quantitatively relate the efficiency of intelligence and command and control systems to firepower and numerical strength

$$\begin{cases} \dfrac{dx}{dt} = -a\left\{\dfrac{xy}{x_0 - e_Y(x_0-x)}\right\} & \text{with} \quad x(0) = x_0, \\[2ex] \dfrac{dy}{dt} = -b\left\{\dfrac{xy}{y_0 - e_X(x_0-x)}\right\} & \text{with} \quad y(0) = y_0, \end{cases}$$

where a and b denote constant LANCHESTER attrition-rate coefficients and e_X and e_Y denote constants that are called the "command efficiencies" of the X and Y forces, respectively. Here both e_X and

$e_Y \in [0,1]$. It should be noted that for "perfect" command efficiency for the Y force (i.e. $e_Y = 1.0$) the X force undergoes attrition at a rate proportional to only the number of enemy firers, while for $e_Y = 0$ this attrition rate is proportional to the product of the numbers of firers and targets. What is the state equation for SCHREIBER's LANCHESTER-type model given by?

19. Consider the following HELMBOLD-type equations for combat between two homogeneous forces in which the larger force suffers inefficiencies of scale when force sizes are grossly unequal.

$$\begin{cases} \dfrac{dx}{dt} = -a\left(\dfrac{x}{y}\right)^{1-W} y & \text{with } x(0) = x_0 , \\[2ex] \dfrac{dy}{dt} = -b\left(\dfrac{y}{x}\right)^{1-W} x & \text{with } y(0) = y_0 , \end{cases}$$

where W denotes a constant and $W \in [0,1]$. What is the state equation for the above LANCHESTER-type combat model given by?

20. The model of the preceding problem treats both forces symmetrically with respect to their inefficiencies of scale in producing casualties in combat operations. Consider now the apparently less symmetric form for such combat with inefficiencies of scale for the larger force

$$\begin{cases} \dfrac{dx}{dt} = -a\left(\dfrac{x}{y}\right)^{1-d} y & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -b\left(\dfrac{y}{x}\right)^{1-e} x & \text{with } y(0) = y_0 , \end{cases}$$

where d and e are constants satisfying $0 \le d,\ e \le 1$. What is the state equation for the above LANCHESTER-type combat model given by? How do you account for the complete symmetry between the two opposing forces in this state equation?

21. Consider a skirmish between homogeneous X and Y forces in which the X force is supported by artillery which delivers area fire against the Y force. This artillery is out of the firing range of the Y force, and hence it suffers no attrition. The Y force withdraws at a constant rate W. Assume that the following LANCHESTER-type equations model the attrition process for this engagement (for x and $y > 0$)

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = x_0 , \\ \\ \dfrac{dy}{dt} = -bx - \alpha y - W & \text{with } y(0) = y_0 , \end{cases}$$

where a, b, α, and W are all positive constants. Assume that x and $y > 0$. What is the Y force level $y(t)$ given by?

22. Consider the following LANCHESTER-type equations for "two-versus-one" aimed-fire combat

$$\begin{cases} \dfrac{dx_1}{dt} = -a_1 y & \text{with } x_1(0) = x_1^0, \\ \dfrac{dx_2}{dt} = -a_2 y & \text{with } x_2(0) = x_2^0, \\ \dfrac{dy}{dt} = -b_1 x_1 - b_2 x_2 & \text{with } y(0) = y_0, \end{cases}$$

where a_1, a_2, b_1, and b_2 are positive constants.

Part a. What is the Y force level $y(t)$ given by?

Part b. Show that the state equation for the above LANCHESTER-type model is given by $z_0^2 - z^2 = (a_1 b_1 + a_2 b_2)(y_0^2 - y^2)$, where $z = b_1 x_1 + b_2 x_2$.

We will now generalize the above results by considering the following LANCHESTER-type equations for "n-versus-one" aimed-fire combat

$$\begin{cases} \dfrac{dx_i}{dt} = -a_i y & \text{with } x_i(0) = x_i^0 \text{ for } i = 1,2,\dots n, \\ \dfrac{dy}{dt} = -\sum\limits_{k=1}^{n} b_k x_k & \text{with } y(0) = y_0, \end{cases}$$

where a_i and b_i for $i = 1,2,\dots,n$ are positive constants.

Part c. What is the state equation for "n-versus-one" combat?

Part d. For "n-versus-one" combat, what is the X_i force level $x_i(t)$ given by?

23. Consider the following LANCHESTER-type equations for aimed-fire combat between two homogeneous forces with superimposed effects of supporting fires that are not subject to attrition (see Figure 2.15 above)

$$\begin{cases} \dfrac{dx}{dt} = -ay - \beta x & \text{with } x(0) = x_0, \\[2ex] \dfrac{dy}{dt} = -bx - \alpha y & \text{with } y(0) = y_0, \end{cases}$$

where a, b, α, and β are all positive constants. Assume that x and $y > 0$.

Part a. What is the X force level $x(t)$ given by?

Part b. What equation is satisfied by the force ratio $u = x/y$?

24. Consider S. DEITCHMAN's [22] LANCHESTER-type model

$$\left\{\begin{aligned} \frac{dx}{dt} &= -ay \qquad \text{with} \quad x(0) = x_0, \\ \frac{dy}{dt} &= -bxy \qquad \text{with} \quad y(0) = y_0, \end{aligned}\right. \tag{I}$$

for the ambush of a homogeneous X counterinsurgent force by a homogeneous Y guerrilla force. Here an individual ambushee returns area fire against aimed fire of the ambushers, since he is "caught in the open by surprise" and only aware of the general region occupied by the ambushers. Consider only that phase of the engagement during which x and $y > 0$.

Part a. Combine the above two LANCHESTER-type equations (I) to obtain a single second-order nonlinear differential equation for the X force level $x(t)$.

Part b. Integrate the second-order equation obtained in Part a to obtain a first-order monlinear differential equation for $x(t)$, i.e. an equation involving only the X force level $x(t)$ for the rate of change of the X force level $\frac{dx}{dt}(t)$.

Part c. Integrate the first-order equation obtained in Part b to obtain the X force level $x(t)$.

FOOTNOTES for Chapter 2

1. H. K. WEISS [101] has pointed out that LANCHESTER, an Englishman, was anticipated (in qualitative but not quantitative terms) in 1905 by BRADLEY A. FISKE (then Commander, but later Rear Admiral, USN), an American. For a sketch of the life and accomplishments of BRADLEY ALLEN FISKE (1854-1942), see [66, pp. 298-299]. J. ENGEL [25] subsequently showed that FISKE's verbal model is equivalent to a system of difference equations (in contrast to LANCHESTER's differential equations) and examined some of the mathematical consequences of these Fiske-type equations of warfare. See Section 2.10 for further details.

2. FREDERICK W. LANCHESTER (1868-1946) was a leading English automotive and aeronautical engineer. In his lifetime, LANCHESTER won the highest honors that his associates could award him [60]: Fellow of the Royal Society, Honorary Doctor of Laws, Honorary Member of the Institution of Mechanical Engineers, Honorary Member and President (1910) of the Institution of Automotive Engineers, and Honorary Fellow of the Royal Aeronautical Society; recipient of the Gold Medal of the Royal Aeronautical Society (1926), of the Daniel Guggenheim Medal (1931), of the Ewing Medal of the Institution of Civil Engineers (1941), and of the James Watt International Medal of the Institution of Mechanical Engineers (1945). For further information about his many scientific and engineering contributions, see McCLOSKEY [60]. In recognition of LANCHESTER's pioneering 1914 contribution [55] (also, again see [60]), which elegantly used mathematical methods for developing insights into the solution of operational problems

long before the term "operations research" was coined, the Operations Research Society of America annually awards the Lanchester Prize "for the paper on operations research judged to be the best of the calendar year."

3. The influential 19th-century German military philosopher, Carl von Clausewitz (1780-1831), stated in his classic work On War (Vom Kriege) [21, p. 276], "The best strategy is always to be very strong, first generally then at the decisive point. . . . There is no more imperative and no simpler law for strategy than to keep the forces concentrated."

4. However, such analytical models may be enriched in detail to become useful operational models through the inclusion of additional state variables, use of more complicated functional relationships between model parameters, etc. (see, for example, W. T. MORRIS [63] for further discussion of the process of such enrichment). Examples of such enriched models that have been used for defense planning are BONDER/IUA, DIVOPS, VECTOR-2, etc. (see Section 1.3).

5. C. ANCKER [1] has pointed out that in 1832 KARL von CLAUSEWITZ [21, p. 101] said that "war is nothing but a duel on an extensive scale."

6. LANCHESTER [55, p. 422] did point out, however, that there were some situations in ancient warfare in which concentration was advantageous.

7. It is still worthwhile to read LANCHESTER's lucid verbal description of combat. The most accessible source is probably MORSE and KIMBALL [64, p. 64] (<u>see</u> also NEWMAN [67, pp. 2138-2140] or, of course, LANCHESTER's original paper [55, column 1 of p. 422]).

8. However, the appropriate equations for such ancient warfare appear in MORSE and KIMBALL [64, p. 65] (<u>see</u> also DOLANSKY [23, p. 346]). These equations are

$$\begin{cases} dx/dt = -1/(1 + E) & \text{with } x(0) = x_0, \\ dy/dt = -E/(1 + E) & \text{with } y(0) = y_0, \end{cases}$$

where all symbols are as defined in the main text.

9. Such an examination does not appear in LANCHESTER's [55] original paper or elsewhere.

10. It should be noted, however, that the concept of equality of fighting strengths must be operationally defined, and such a definition invariably involves a model of battle termination, i.e. the specifications of "victory" and "draw" conditions. With this in mind, we observe that LANCHESTER (implicitly) developed (2.1.6) for a "fight-to-the-finish," and the condition for equality of fighting strengths must be modified in other cases (<u>see</u> Section 2.8 and Chapter 3).

11. In fact, LANCHESTER [55] did not develop (2.1.5) at all. Equation (2.1.5) was apparently first given by MORSE and KIMBALL [64, p. 65]

and called "LANCHESTER's square law" by them.

12. In his original 1914 paper [55], LANCHESTER did not explicitly give the force-ratio equation (2.1.7) in his development of the "square law" (2.1.6), but he enigmatically determined conditions under which $(1/x)dx/dt - (1/y)dy/dt - (1/u)du/dt = 0$. Thus, LANCHESTER himself only implicitly considered the force-ratio equation (2.1.7) in the development of his famous square law (2.1.6).

13. In modelling combat with two such differential equations for the two force levels, one is implicitly assuming that the force levels are the state variables, i.e the future course of combat may be predicted from knowledge of only the current values of the force levels (assuming that the attrition-rate coefficients a and b are known) (see Section 1.6 above). There is, moreover, far from universal agreement as to what are the significant (i.e. state) variables for modelling military combat. For some other views, see HAYWARD [30] or LIDDELL HART [56].

14. Corresponding stochastic combat formulations (i.e. MARKOV-chain analogues) are for all practical purposes analytically intractable. Furthermore, very nearly the same trends for the combat dynamics are obtained from deterministic and corresponding stochastic models although some caution must be exercised in considering only the deterministic model for small numbers of combatants or when the forces are "near parity" (see Chapter 4 below). Moreover, BONDER and FARRELL [11] have reported excellent agreement between Monte Carlo or stochastic simulation results and those for a corresponding deterministic LANCHESTER-type model.

15. Initially, we were tempted to call (2.2.1) "LANCHESTER's equations for a 'square-law' attrition process," since they do yield the quadratic state equation (2.2.5) (see TAYLOR [82; 84]). However, there are many differential combat models besides (2.2.1) that yield (2.2.5) (see Section 2.9 below). Consequently, we have chosen the name "LANCHESTER's equations for modern warfare," although the equations (2.2.1) have been hypothesized to apply under other conditions. Sometimes it will be more convenient to refer to (2.2.1) as a F|F LANCHESTER-type attrition process (or, simply, F|F attrition) when greater preciseness is required (see Section 2.12).

16. Of course, the exact information to be extracted from a model (even a simple one) depends on the purpose of the study under consideration.

17. Except for the special case of quasi-autonomous equations in which case the equations may be transformed to constant-coefficient ones by a change of the time scale (see Section 6.3 below).

18. Actually, if we recall (2.2.2), the X force level is given by

I. when $x_0/y_0 = \sqrt{a/b}$:

$$x(t) = x_0 e^{-\sqrt{ab}\, t} \qquad \text{for } 0 \le t \le +\infty ,$$

II. when $x_0/y_0 < \sqrt{a/b}$:

$$x(t) = \begin{cases} x_0 \cosh \sqrt{ab}\, t - y_0\sqrt{\frac{a}{b}} \sinh \sqrt{ab}\, t & \text{for } 0 \le t \le t_X^a , \\ 0 & \text{for } t \ge t_X^a , \end{cases}$$

where

$$t_X^a = \frac{1}{2\sqrt{ab}} \ln\left(\frac{\sqrt{a}\, y_0 + \sqrt{b}\, x_0}{\sqrt{a}\, y_0 - \sqrt{b}\, x_0}\right) ,$$

III. when $x_0/y_0 > \sqrt{a/b}$:

$$x(t) = \begin{cases} x_0 \cosh \sqrt{ab}\, t - y_0\sqrt{\frac{a}{b}} \sinh \sqrt{ab}\, t & \text{for } 0 \le t \le t_Y^a , \\ x_0 \sqrt{1 - (a/b)(y_0/x_0)^2} & \text{for } t \ge t_Y^a , \end{cases}$$

where

$$t_Y^a = \frac{1}{2\sqrt{ab}} \ln\left(\frac{\sqrt{b}\, x_0 + \sqrt{a}\, y_0}{\sqrt{b}\, x_0 - \sqrt{a}\, y_0}\right) .$$

It will be convenient in subsequent developments to relax the requirement that $x, y \ge 0$.

19. To see this, consider the solution to (2.2.1) for $t \ge t_a > 0$ with intermediate condition $x(t_a) = 0$ and $y(t_a) = 0$ but $x_0 y_0 \neq 0$. Clearly, $x = y \equiv 0$ is a solution to (2.2.1). By a standard uniqueness theorem, it is the solution, and we must have $x_0 = y_0 = 0$, which is a contradiction. Hence, it is impossible to have both $x(t)$ and $y(t)$ equal to zero at any finite time if $x_0 y_0 \neq 0$.

20. Of course, the easiest way to determine $u(t)$ is to form the ratio $x(t)/y(t)$ with $x(t)$ given by (2.2.13) and $y(t)$ given by (2.2.15).

21. Or, equivalently, a quasi-autonomous model, i.e. one that may be transformed into a constant-coefficient model by a transformation of the battle's time scale.

22. Initially, were tempted to call (2.4.1) "LANCHESTER's equations for a 'linear-law' attrition process," since they do yield the linear state equation (2.4.3) (see TAYLOR [82; 84]). However, there are many differential combat models besides (2.4.1) that yield (2.4.3) (see Section 2.9 below). Consequently, we have chosen the name "LANCHESTER's equations for area fire," although the equations (2.4.1) have been hypothesized to apply under other conditions. Sometimes it will be more convenient to refer to (2.4.1) as a FT|FT LANCHESTER-type attrition process (or, simply, FT|FT attrition) when greater preciseness is required (see Section 2.12).

23. Namely, the class of differential equations of the form

$$\frac{d^2w}{dz^2} = F(z,w,w') \quad ,$$

where F is rational in w and w', and analytic in z, which have all their critical points (i.e. branch points and essential singularities) fixed (see INCE [41, p. 335]).

24. We again caution the reader that the attrition-rate coefficients a and b, however, represent different physical quantities in the two models (2.2.1) and (2.4.1).

25. In general, we have

$$\frac{\frac{du}{dt}}{\{-\frac{1}{y}\frac{dy}{dt}\}} = u - \frac{dx}{dy} \quad ,$$

which (assuming that $dy/dt < 0$) shows that the difference between the force ratio u and the differential force-change ratio (for cases of no replacements and withdrawals, the differential casualty-exchange ratio) dx/dy determines the sign of the rate of change of the force ratio (see TAYLOR [89]).

26. For example, a tank designer might be interested in developing an explicit tradeoff between certain performance parameters of a tank weapon system or between different tanks (e.g. weight of armor [i.e. degree of protection] versus mobility for a tank). Moreover, although a simplified analytical model may well be far too simple to be able to solve by itself such an operational problem (i.e. neither assert what decisions should be made nor predict what decisions will be made), it may be quite useful in exposing the bare determinants of the tradeoff or decision (i.e. identifying the major factors and developing a rough quantitative relationship between them). See SHUBIK and BREWER [74] and PAXSON [68, p. 8] for further discussions.

27. Here we are using the word uncertainty in a nontechnical sense [as opposed to the usual technical sense in which the word is used in OR (see for example, LUCE and RAIFFA [58, p. 13])].

28. As pointed out by TAYLOR and PARRY [95], the entire subject of modelling battle termination is a problem area in contemporary defense-planning studies. There is far from universal agreement on this topic (see TAYLOR [83] and also Chapter 3 for further references).

29. This distinction was kindly pointed out to the author by the referees to his paper TAYLOR [91].

30. As emphasized in TAYLOR [91], it will not make much sense to study decisions under uncertainty unless we know how to make decisions under full certainty.

31. In reality, however, the actual trend in combat operations over the past two thousand years of military history (see [40]) has been towards greater dispersion of forces (i.e. lower troop density). We will discuss this point further below in Chapter 6 and will explain why it is so with another equally simple model.

32. For the adopted battle-termination conditions (each side has a fixed breakpoint), enemy casualties have been fixed, and consequently it was not necessary to consider them. In other cases, however, the victor might very well want to also consider enemy losses in his force-concentration decision.

33. FISKE essentially developed his own version of firepower scores for naval engagements. This was, of course, done long before the term "firepower score" was coined.

34. Clearly, the equations are physically meaningful only for $x_n, y_n > 0$. Strictly speaking thus, we should adopt some convention like (2.2.2), but for simplicity have omitted this.

35. As shown in Chapter 7, we can always achieve this condition by taking the length of the time period to be short enough.

36. See, for example, SOKOLINKOFF [79, p. 51 and p. 263] or BUNGE [18, p. 349] (also LINDSAY and MARGENAU [57]).

37. G. CLARK [20] has emphasized that one may consider target-acquisition capability as the distinguishing characteristic between the two sets of physical circumstances (i.e. those given in Tables 2.XVIII and 2.XIX) that have been hypothesized to yield these two different basic combat models. When targets are readily acquired, the modern-warfare equations (2.2.1) arise; while when only general knowledge of target locations is available, the area-fire equations (2.4.1) arise. However, CLARK [20; p. 19] erroneously attributes this observation to LANCHESTER [55]. It apparently is due to WEISS [99].

38. For example, Taylor and Brown [92] and Taylor and Comstock [94] show that the representation of solutions [92] and the development of force-annihilation-prediction conditions [94] for variable-coefficient LANCHESTER-type equations of modern warfare may be considered to be a generalization of these constant-coefficient results.

39. MORSE and KIMBALL [64, p. 71] originally did not use the term "operational" losses in this sense: they used it to denote war losses under operating conditions. MORSE and KIMBALL considered a simple model for the overall trend of a war and hypothesized that besides a term of the form ay in

X's operational loss rate, i.e (-dx/dt), there should be one proportional to X's size, i.e. βx. Apparently, this terminology was later changed by BACH, DOLANSKY, and STUBBS [3] to denote losses not due to enemy action.

40. Although not explictly stated in his paper [36], HELMBOLD apparently based his modification of LANCHESTER's equations for modern warfare (2.2.1) on the results of rather extensive empirical investigations (<u>see</u> HELMBOLD [32-34]). His basic idea for these investigations was to find regularities or "patterns" in historical battle data and then to determine whether or not a given simple combat models (HELMBOLD took LANCHESTER's equations of modern warfare) exhibits a similar "pattern." From his historical data, HELMBOLD found that relative fire effectiveness a/b (i.e. the ratio of the fire effectiveness of an individual Y combatant to that of an individual X combatant) to be strongly correlated with the initial force ratio x_0/y_0.

HELMBOLD's data base consisted of initial and final force levels for both sides for several hundred historical battles, with one side identified as the attacker (X) and the other (Y) as the defender. Assuming that the square law (2.2.5) held, HELMBOLD computed the initial force ratio x_0/y_0, survivor fractions x_f/x_0 and y_f/y_0, advantage parameter $V = \ln \mu$ where $\mu = \{1 - (x_f/x_0)^2\}/\{1 - (y_f/y_0)^2\}$, activity ratio (in our terminology, relative fire effectiveness) a/b, and the "bitterness" parameter $\varepsilon = \sqrt{ab}\, t_f$ for each of (all told for the three investigations [32 - 34]) several hundred battles. As indicated above, his idea was to collect a sizable body of data dealing with the historical

battles, use this data to compute parameters (advantage, activity ratio, and bitterness) associated with each battle, and search the results for regularities. Although we would expect the initial force ratio x_0/y_0 and the relative fire effectiveness a/b to be independent parameters, in carrying out the above program, HELMBOLD found them to be strongly positively correlated (<u>see</u>, in particular, [32, p. 7; 33, pp. 31-35 and 58-59]).

Thus, if one assumes that the square law (2.2.5) holds, then available historical battle data says that x_0/y_0 and a/b are strongly positively correlated: as the initial force ratio of X to Y increases, the relative fire effectiveness of an individual X combatant to that of an individual Y one decreases. Thus, one is led to abandon the model (2.2.1) and to conjecture that the larger force suffers inefficiencies of scale when force sizes are grossly unequal. HELMBOLD's model (2.12.6) is a mathematical expression of this hypothesis.

41. Although not stated by HELMBOLD [36], this particular functional form is suggested by his linear regression results for $\ln a/b$ against $\ln x_0/y_0$ (<u>see</u> [33; 34]). <u>See</u> also above footnote.

42. This model is particularly important because an extension of it can be used to model the casualty-rate curves used in several theater-level combat models (<u>see</u> Chapter 8).

43. The greater convenience and insight to be gained by introducing the parameter δ was apparently first observed by H. K. WEISS [103].

REFERENCES for Chapter 2

1. C. J. Ancker, "The Status of Developments in the Theory of Stochastic Duels - II," Opns. Res. 15, 388-406 (1967).

2. L. B. Anderson, J. Bracken, and E. Schwartz, "Revised OPTSA Model," P-1111, Institute for Defense Analyses, Arlington, Virginia, September 1975.

3. R. Bach, L. Dolansky, and H. Stubbs, "Some Recent Contributions to the Lanchester Theory of Combat," Opns. Res. 10, 314-326 (1962).

4. C. Barfoot, "The Lanchester Attrition-Rate Coefficient: Some Comments on Seth Bonder's Paper and a Suggested Alternate Method," Opns. Res. 17, 888-894 (1969).

5. R. Bellman and S. Dreyfus, "On a Tactical Air-Warfare Model of Mengel," Opns. Res. 6, 65-78 (1958).

6. L. Berkovitz and M. Dresher, "A Game Theory Analysis of Tactical Air War," Opns. Res. 7, 599-620 (1959).

7. L. Berkovitz and M. Dresher," Allocation of Two Types of Aircraft in Tactical Air War: A Game Theoretic Analysis," Opns. Res. 8, 694-706 (1960).

8. S. Bonder, "Combat Model," Chapter 2 in "The Tank Weapon System," S. Bonder and D. Howland (Editors), Report No. RF 573 AR 64-1, Systems Research Group, The Ohio State University, Columbus, Ohio, June 1964 (AD 447 494).

9. S. Bonder, "The Lanchester Attrition-Rate Coefficient," Opns. Res. 15, 221-222 (1967).

10. S. Bonder, "The Mean Lanchester Attrition Rate," Opns. Res. 18, 179-181 (1970).

11. S. Bonder and R. Farrell (Editors), "Development of Models for Defense Systems Planning," Report No. SRL 2147 TR 70-2, Systems Research Laboratory, The University of Michigan, Ann Arbor, Michigan, September 1970.

12. S. Bonder and J. Honig, "An Analytic Model of Ground Combat: Design and Application," Proc. U.S. Army Opns. Res. Symp. 10, 319-394 (1971).

13. J. Bracken, "Two Optimal Sortie Allocation Models, Volume I: Methodology and Sample Results," P-992, Institute for Defense Analyses, Arlington, Virginia, December 1973.

14. J. Bracken, J. Falk, and A. Karr, "Two Models for Optimal Allocation of Aircraft Sorties," Opns. Res. 23, 979-995 (1975).

15. H. Brackney, "The Dynamics of Military Combat," Opns. Res. 7, 30-44 (1959).

16. R. Brown, "A Stochastic Analysis of Lanchester's Theory of Combat," ORO-T-323, Operations Research Office, The Johns Hopkins University, Chevy Chase, Maryland, December 1955 (AD 82 944).

17. R. Brown, "Theory of Combat: The Probability of Winning," Opns. Res. 11, 418-425 (1963).

18. M. Bunge, Scientific Research, Volume I: The Search for System, Springer-Verlag New York, Inc., New York, 1967.

19. W. P. Cherry, "The Role of Differential Models of Combat in Fire Support Analyses," Appendix 4 in Fire Support Requirements Methodology Study Phase II, Proceedings of the Fire Support Methodology Workshop, R. M. Thackeray (Editor), Ketron, Inc., Arlington, Virginia, December 1975.

20. G. Clark, "The Combat Analysis Model," Ph.D. Thesis, The Ohio State University, Columbus, Ohio, 1969.

21. C. von Clausewitz, On War, edited with an introduction by A. Rapoport, Penguin Books Ltd., Harmondsworth, Middlesex, England, 1968.

22. S. Deitchman, "A Lanchester Model of Guerrilla Warfare," Opns. Res. 10, 818-827 (1962).

23. L. Dolansky, "Present State of the Lanchester Theory of Combat," Opns. Res. 12, 344-358 (1964).

24. J. Engel, "A Verification of Lanchester's Law," Opns. Res. 2, 163-171 (1954).

25. J. Engel, "Comments on a Paper by H. K. Weiss," Opns. Res. 11, 147-150 (1963).

26. R. Farrell, "VECTOR 1 and BATTLE: Two Versions of a High-Resolution Ground and Air Theater Campaign Model," pp. 233-241 in Military Strategy and Tactics, R. Huber, L. Jones, and E. Reine (Editors), Plenum Press, New York, 1975.

27. D. Fulkerson and S. Johnson, "A Tactical Air Game," Opns. Res. 5, 704-712 (1957).

28. L. Giamboni, A. Mengel, and R. Dishington, "Simplified Model of a Symmetric Tactical Air War," RM-711, The RAND Corporation, Santa Monica, California, August 1951.

29. F. Grubbs and J. Shuford, "A New Formulation of Lanchester Combat Theory," Opns. Res. 21, 926-941 (1973).

30. P. Hayward, "The Measurement of Combat Effectiveness," Opns. Res. 16, 314-323 (1968).

31. O. Hellman, "An Extension of Lanchester's Linear Law," Opns. Res. 14, 931-935 (1966).

32. R. Helmbold, "Lanchester Parameters for Some Battles of the Last Two Hundred Years," CORG-SP-122, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, February 1961 (AD 481 201).

33. R. Helmbold, "Historical Data and Lanchester's Theory of Combat," CORG-SP-128, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, July 1961 (AD 480 975).

34. R. Helmbold, "Historical Data and Lanchester's Theory of Combat, Part II," CORG-SP-190, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, August 1964 (AD 480 109).

35. R. Helmbold, "Some Observations on the Use of Lanchester's Theory for Prediction," Opns. Res. 12, 778-781 (1964).

36. R. Helmbold, "A Modification of Lanchester's Equations," Opns. Res. 13, 857-859 (1965).

37. R. Helmbold, "Decision in Battle: Breakpoint Hypotheses and Engagement Termination Data," R-772-PR, The RAND Corporation, Santa Monica, California, June 1971.

38. E. Hille, Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, Massachusetts, 1969.

39. E. Hille, Ordinary Differential Equations in the Complex Domain, John Wiley, New York, 1976.

40. Historical Evaluation and Research Organization (HERO), "The Fundamentals of Land Combat for Developing Computer Simulation Models of Ground and Air-Ground Warfare," unpublished seminar notes, Dunn Loring, Virginia, 1976.

41. E. Ince, Ordinary Differential Equations, Longmans, Green and Co., London, 1927 (reprinted by Dover Publications, Inc., New York, 1956).

42. R. Isaacs, "Differential Games I: Introduction, "RM-1391, The RAND Corporation, Santa Monica, California, November 1954.

43. R. Isaacs, "Differential Games IV: Mainly Examples," RM-1486, The RAND Corporation, Santa Monica, California, March 1955.

44. R. Isaacs, Differential Games, John Wiley, New York, 1965.

45. J. Isbell and W. Marlow, "Attrition Games," Naval Res. Log. Quart. 3, 71-94 (1956).

46. J. Isbell and W. Marlow, "Methods of Mathematical Tactics," Logistics Papers, No. 14, The George Washington University Logistics Research Project, Washington, D.C., September 1956.

47. C. W. Karns, "An Application of Lanchester's Equations to Amphibious Assaults," ORG Study No. 1, ORG 126-53, Operations Research Group, Office of Naval Research, December 1953 (AD 38 491).

48. A. Karr, "Stochastic Attrition Models of Lanchester Type," P-1030, Institute for Defense Analyses, Arlington, Virginia, June 1974.

49. A. Karr, "On Simulations of the Stochastic, Homogeneous, Lanchester Square-Law Attrition Processes," P-1112, Institute for Defense Analyses, Arlington, Virginia, September 1975.

50. A. Karr, "On Simulations of the Stochastic, Homogeneous, Lanchester Linear-Law Attrition Process," P-1113, Institute for Defense Analyses, Arlington, Virginia, September 1975.

51. A. Karr, "A Class of Lanchester Attrition Processes," P-1230, Institute for Defense Analyses, Arlington, Virginia, December 1976.

52. Y. Kawara, "An Allocation Problem of Fire Support in Combat as a Differential Game," Opns. Res. 21, 942-951 (1973).

53. S. Kimbelton, "Attrition Rates for Weapons with Markov-Dependent Fire," Opns. Res. 19, 698-706 (1971).

54. T. Kisi and T. Hirose, "Winning Probability in an Ambush Engagement," Opns. Res. 14, 1137-1138 (1966).

55. F. W. Lanchester, "Aircraft in Warfare: The Dawn of the Fourth Arm - No. V., The Principle of Concentration," Engineering 98, 422-423 (1914) (reprinted on pp. 2138-2148 of The World of Mathematics, Vol. IV, J. Newman (Editor), Simon and Schuster, New York, 1956).

56. B. H. Liddell Hart, Strategy, Frederick A. Praeger, Inc., New York, 1967 (Second Revised Edition).

57. R. Lindsay and H. Margenau, Foundations of Physics, John Wiley, New York, 1936 (reprinted by Dover Publications, Inc., New York, 1957).

58. R. Luce and H. Raiffa, Games and Decisions, John Wiley, New York, 1957.

59. A. Maradudin and G. Weiss, "A Study of Some Lanchester-Like Equations," Tech. Report No. 95, Physics Dept., University of Maryland, College Park, Maryland, 1958 (AD 154 168).

60. J. McCloskey, "Of Horseless Carriages, Flying Machines, and Operations Research: A Tribute to Frederick William Lanchester (1868-1946)," Opns. Res. 4, 141-147 (1956).

61. A. Mengel, "Optimum Tactics in an Air Superiority Campaign," RM-1068, The RAND Corporation, Santa Monica, California, April 1953.

62. A. Mengel, "Optimization in Dynamic Allocation Problems by a Modified Calculus of Variations Technique," RM-1379, The RAND Corporation, Santa Monica, California, September 1954.

63. W. T. Morris, "On the Art of Modelling," Management Sci. 13, B-707 - B-717 (1967).

64. P. Morse and G. Kimball, Methods of Operations Research, The M.I.T. Press, Cambridge, Massachusetts, 1951.

65. R. Mulholland and R. Specht, "The Rate of Advance of the Front Line in Some World War II Campaigns," RM-1072 The RAND Corporation, Santa Monica, California, April 1953.

66. The National Cyclopaedia of American Biography, Vol. 31, James F. White and Co., New York, 1944.

67. J. R. Newman (Editor), The World of Mathematics, Vol. IV, Simon and Schuster, New York, 1956.

68. E. W. Paxson, "War Gaming," RM-3489-PR, The RAND Corporation, Santa Monica, California, February 1963.

69. R. Peterson, "Methods of Tank Combat Analysis," pp. 134-150 in "Report of Fifth Tank Conference Held at Aberdeen," H. Goldman and G. Zeller (Eds.), Report No. 918, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, July 1953 (AD 46 000).

70. R. Peterson, "On the 'Logarithmic Law' of Attrition and Its Application to Tank Combat," Opns. Res. 15, 557-558 (1967).

71. V. Savkin, The Basic Principles of Operational Art and Tactics, Moscow, 1972 (translated by the U.S. Air Force and published by the U.S. Government Printing Office, Washington, D.C., 1974).

72. M. Schaffer, "Lanchester Models of Guerrilla Engagements," Opns. Res. 16, 457-488 (1968).

73. T. Schreiber, "Note on the Combat Value of Intelligence and Command Control Systems," Opns. Res. 12, 507-510 (1964).

74. M. Shubik and G. Brewer, "Models, Simulations, and Games - A Survey," R-1060-ARPA/RC, The RAND Corporation, Santa Monica, California, May 1972.

75. J. Shuford and F. Grubbs, "Stopping Rules for War Games or Combat Simulations with Exponential Life-Times," Opns. Res. 23, 824-829 (1975).

76. C. Siska, L. Giamboni, and J. Lind, "Analytic Formulation of a Theater Air-Ground Warfare System (1953 Techniques)," RM-1338, The RAND Corporation, Santa Monica, California, September 1954.

77. D. Smith, "The Probability Distribution of the Number of Survivors in a Two-Sided Combat Situation," Operational Res. Quart. 16, 429-437 (1965).

78. R. Snow, "Contributions to Lanchester Attrition Theory," Report RA-15078, The RAND Corporation, Santa Monica, California, April 1948.

79. I. S. Sokolnikoff, Tensor Analysis, John Wiley, New York, 1951.

80. A. Springall, "Contributions to Lanchester Combat Theory," Ph.D. Thesis, Virginia Polytechnic Institute, Blacksburg, Virginia, March 1968.

81. J. Taylor, "A Note on the Solution to Lanchester-Type Equations with Variable Coefficients," Opns. Res. 19, 709-712 (1971).

82. J. Taylor, "Target Selection in Lanchester Combat: Linear Law Attrition Process," Naval Res. Log. Quart. 20, 673-697 (1973).

83. J. Taylor, "Survey on the Optimal Control of Lanchester-Type Attrition Processes," presented at the Symposium on the State-of-the-Art of Mathematics in Combat Models, June 1973 (also Tech. Report NPS55TW74031, Naval Postgraduate School, Monterey, California, March 1974) (AD 778 630).

84. J. Taylor, "Lanchester-Type Models of Warfare and Optimal Control," Naval Res. Log. Quart. 21, 79-106 (1974).

85. J. Taylor, "Some Differential Games of Tactical Interest," Opns. Res. 22, 304-317 (1974).

86. J. Taylor, "Solving Lanchester-Type Equations for 'Modern Warfare' with Variable Coefficients," Opns. Res. 22, 756-770 (1974).

87. J. Taylor, "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Naval Res. Log. Quart. 21, 683-704 (1974).

88. J. Taylor, "On the Treatment of Force-Level Constraints in Time-Sequential Combat Problems," Naval Res. Log. Quart. 22, 617-650 (1975).

89. J. Taylor, "On the Relationship Between the Force Ratio and the Instantaneous Casualty-Exchange Ratio for Some Lanchester-Type Models of Warfare," Naval Res. Log. Quart. 23, 345-352 (1976).

90. J. Taylor, "Determining the Class of Payoffs that Yield Force-Level-Independent Optimal Fire-Support Strategies," Opns. Res. 25, 506-516 (1977).

91. J. Taylor, "Optimal Commitment of Forces in Some Lanchester-Type Combat Models," Opns. Res. 27, 96-114 (1979).

92. J. Taylor and G. Brown, "Canonical Methods in the Solution of Variable-Coefficient Lanchester-Type Equations of Modern Warfare," Opns. Res. 24, 44-69 (1976).

93. J. Taylor and G. Brown, "An Examination of the Effects of the Criterion Functional on Optimal Fire-Support Policies," Naval Res. Log. Quart. 25, 183-211 (1978).

94. J. Taylor and C. Comstock, "Force-Annihilation Conditions for Variable-Coefficient Lanchester-Type Equations of Modern Warfare," Naval Res. Log. Quart. 24, 349-371 (1977).

95. J. Taylor and S. Parry, "Force-Ratio Considerations for Some Lanchester-Type Models of Warfare," Opns. Res. 23, 522-533 (1975).

96. R. Watson, "An Application of Martingale Methods to Conflict Models," Opns. Res. 24, 380-382 (1976).

97. G. Weiss, "Comparison of a Deterministic and a Stochastic Model for Interaction Between Antagonistic Species," Biometrics 19, 595-602 (1963).

98. H. K. Weiss, "Requirements for a Theory of Combat," Memorandum Report No. 667, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, April 1953 (AD 13 717).

99. H. K. Weiss, "Lanchester-Type Models of Warfare," pp. 82-98 in Proc. First International Conference on Operational Research, M. Davies, R. Eddison, and T. Page (Editors), Operations Research Society of America, Baltimore, Maryland, 1957.

100. H. K. Weiss, "Some Differential Games of Tactical Interest and the Value of a Supporting Weapon System" Opns. Res. 7, 180-196 (1959).

101. H. K. Weiss, "The Fiske Model of Warfare," Opns. Res. 10, 569-571 (1962).

102. H. K. Weiss, "Combat Models and Historical Data: The U.S. Civil War," Opns. Res. 14, 759-790 (1966).

103. H. K. Weiss, "Review of Lanchester Models of Warfare," presented at 30th National ORSA Meeting, Durham, North Carolina, October 1966.

104. D. Willard, "Lanchester as Force in History: An Analysis of Land Battles of the Years 1618-1905," RAC-TP-74, Research Analysis Corporation, Bethesda, Maryland, November 1962.

APPENDIX A: BACKGROUND FOR THE MATHEMATICS OF LANCHESTER'S CLASSIC COMBAT FORMULATIONS

Appendix A consists of three parts: A.1 The Hyperbolic Functions, A.2 Solution to the nth Order Constant-Coefficient Linear Differential Equation, and A.3 The Generalized RICCATI Equation. Its purpose is to provide some general mathematical background for Chapter 2, which was not convenient to incorporate into the main text. The reader who is familiar with the hyperbolic functions, solving constant-coefficient linear differential equations, and the RICCATI equation may skip this material.

APPENDIX A.1: The Hyperbolic Functions

1. Background.

The so-called hyperbolic functions are similar to the well-known circular functions (e.g. sine, cosine, etc.). Let us recall that via Euler's formula (here $i = \sqrt{-1}$)

$$e^{i\theta} = \cos\theta + i\sin\theta\,, \qquad \text{(A.1.1)}$$

we may write

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\,, \qquad \text{(A.1.2)}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\,. \qquad \text{(A.1.3)}$$

It has been convenient to introduce into mathematical analysis related functions called the hyperbolic functions, since in many applications the exponential function enters in combinations of the form $\frac{1}{2}(e^{\theta} + e^{-\theta})$ or $\frac{1}{2}(e^{\theta} - e^{-\theta})$. Thus, we introduce the so-called hyperbolic cosine

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}\,, \qquad \text{(A.1.4)}$$

and the hyperbolic sine

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}\,. \qquad \text{(A.1.5)}$$

Replacing θ by $i\theta$ in (A.1.2) and (A.1.3), we see that $\cos i\theta = \cosh\theta$ and $\sin i\theta = i\sinh\theta$, which provides one motivation for the names hyperbolic cosine and hyperbolic sine.

2. Properties Useful for LANCHESTER Combat Theory.

From the above definitions of the hyperbolic functions, one readily deduces (show this yourself) the following properties:

(P1) $\frac{d}{d\theta} \cosh \theta = \sinh \theta$,

(P2) $\frac{d}{dt} \sinh \sqrt{ab}\, t = \sqrt{ab} \cosh \sqrt{ab}\, t$,

(P3) for $t = 0$, $\cosh \sqrt{ab}\, t = 1$ and $\sinh \sqrt{ab}\, t = 0$,

(P4) $\cosh(u-v) = \cosh u \cosh v - \sinh u \sinh v$,

and (P5) $\sinh(u-v) = \sinh u \cosh v - \cosh u \sinh v$.

Let us now briefly show how the above properties are useful for LANCHESTER combat theory. Properties (P1) and (P2) imply that the general solution to

$$\frac{d^2x}{dt^2} - ab\, x = 0 \, , \qquad \text{(A.1.6)}$$

is given by

$$x(t) = A \cosh \sqrt{ab}\, t + B \sinh \sqrt{ab}\, t. \qquad \text{(A.1.7)}$$

We recall that (A.1.6) is the X force-level equation, which appears in the main text as (2.2.10), so that its initial conditions are

$$x(0) = x_0 \, , \qquad \text{and} \qquad \frac{dx}{dt}(0) = -ay_0 \, . \qquad \text{(A.1.8)}$$

Property (P3) is useful for evaluating the constants A and B in (A.1.7): using (A.1.8), we find that $A = x_0$ and $B = -y_0 \sqrt{a/b}$ so that

$$x(t) = x_0 \cosh \sqrt{ab}\, t - y_0 \sqrt{\frac{a}{b}} \sinh \sqrt{ab}\, t \, , \qquad \text{(A.1.9)}$$

which appears in the main text as (2.2.9). The reader should note the great convenience for evaluating the constants in the general solution to (A.1.6) (see Appendix A.2) when it is expressed in terms of the hyperbolic functions.

Properties (P4) and (P5) are two of the so-called algebraic addition theorems possessed by the hyperbolic functions. If we consider the battle to begin at t_0, then the initial conditions to (A.1.6) are

$$x(t_0) = x_0 , \qquad \text{and} \qquad \frac{dx}{dt}(t_0) = -ay_0 ,$$

whence (A.1.7) yields

$$\begin{aligned} x(t) = x_0(&\cosh \sqrt{ab}\, t_0 \cosh \sqrt{ab}\, t - \sinh \sqrt{ab}\, t_0 \sinh \sqrt{ab}\, t) \\ &+y_0 \sqrt{\frac{a}{b}} (\cosh \sqrt{ab}\, t_0 \sinh \sqrt{ab}\, t - \sinh \sqrt{ab}\, t_0 \cosh \sqrt{ab}\, t). \end{aligned} \tag{A.1.10}$$

The addition theorems (P4) and (P5) yield, however, that (A.1.10) may be simplified to

$$x(t) = x_0 \cosh \sqrt{ab}\,(t-t_0) - y_0 \sqrt{\frac{a}{b}} \sinh \sqrt{ab}\,(t-t_0) . \tag{A.1.11}$$

Let us emphasize that the algebraic addition theorems of the hyperbolic functions are the reason that (A.1.10) may be simplified to (A.1.11).

It is also convenient to introduce the so-called <u>hyperbolic tangent</u> defined by, in analogy with the circular functions,

$$\tanh \theta = \frac{\sinh \theta}{\cosh \theta} . \tag{A.1.12}$$

It may be shown that the hyperbolic tangent has the following properties:

(P6) $\tanh \theta$ is a strictly increasing function of θ, with $\tanh \theta = 0$ for $\theta = 0$,

(P7) $\lim_{t \to +\infty} \tanh \theta = 1$.

If we write the X force level as a function of time as

$$x(t) = \left\{ x_0 - y_0 \sqrt{\frac{a}{b}} \tanh \sqrt{ab}\, t \right\} \cosh \sqrt{ab}\, t \,, \tag{A.1.13}$$

then the terms within the brackets (namely, $F(t) = x_0 - y_0 \sqrt{a/b} \tanh \sqrt{ab}\, t$) determine the sign of $x(t)$, since $\cosh \theta$ is always positive. Observing that $F(t)$ is a strictly decreasing function of time with $F(0) = x_0 > 0$ and $\lim_{t \to +\infty} F(t) = x_0 - y_0 \sqrt{a/b}$, we see from (A.1.13) that $x(t) > 0$ for all $t \geq 0$ if and only if $x_0/y_0 > \sqrt{a/b}$. Conversely, X will be annihilated in finite time if and only if $x_0/y_0 < \sqrt{a/b}$. Furthermore, the time at which X is annihilated, i.e. t_a^X such that $x(t_a^X) = 0$, is given by

$$t_a^X = \frac{1}{\sqrt{ab}} \tanh^{-1}\left(\frac{x_0}{y_0} \sqrt{\frac{b}{a}} \right),$$

which is well defined for $x_0/y_0 < \sqrt{a/b}$ by virtue of (P6) and (P7). What this all shows is that force-annihilation prediction for the model (2.2.1) is intimately related to the properties (P6) and (P7) of the hyperbolic tangent.

Finally, the material in this section is essential for understanding TAYLOR and BROWN's [2] ideas for representing the solution to variable-coefficient LANCHESTER-type equations of modern warfare (see Chapter 6).

For further information about the hyperbolic functions, the reader can consult any good text on the calculus (<u>see</u>, for example, COURANT and JOHN [1, pp. 228-236]).

REFERENCES for Appendix A.1

1. R. Courant and F. John, Introduction to Calculus and Analysis, Volume One, Interscience, New York, 1965.

2. J. Taylor and G. Brown, "Canonical Methods in the Solution of Variable-Coefficient Lanchester-Type Equations of Modern Warfare," Opns. Res. 24, 44-69 (1976).

APPENDIX A.2: Solution to the nth Order Constant-Coefficient Linear Differential Equation

1. General Results.

All force-level equations for LANCHESTER's equations of modern warfare (2.2.1) and its extension to combat between heterogeneous forces (and many other differential combat models) are in one sense or another special cases of the nth order constant-coefficient linear homogeneous differential equation

$$Lx = \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n = 0 , \qquad \text{(A.2.1)}$$

where the coefficients a_k are constants.

We first attempt to determine n linearly independent solutions to (A.2.1). The appearance of this homogeneous equation suggests homogeneous solutions of the form $e^{\rho t}$, where ρ is a constant, since all derivatives of $e^{\rho t}$ are constant multiplies of the function itself, i.e.

$$\frac{d^m}{dt^m} e^{\rho t} = \rho^m e^{\rho t} .$$

We have then

$$Le^{\rho t} = (\rho^n + a_1 \rho^{n-1} + \cdots + a_{n-1}\rho + a_n)e^{\rho t} .$$

This result shows that $e^{\rho t}$ is a solution of (A.2.1) if ρ satisfies the so-called characteristic equation

$$\rho^n + a_1\rho^{n-1} + \cdots + a_{n-1}\rho + a_n = 0 \ . \qquad \text{(A.2.2)}$$

Let us note that the characteristic equation may be obtained from the original homogeneous differential equation (A.2.1) simply by formally replacing d^kx/dt^k by ρ^k, with the convention that $d^0x/dt^0 \equiv x$.

Since (A.2.2) has n roots $\rho_1, \rho_2, \ldots, \rho_n$, it may be written in the form

$$(\rho-\rho_1)(\rho-\rho_2) \cdots (\rho-\rho_n) = 0 \ .$$

If the n roots are distinct, exactly n linearly independent solutions $e^{\rho_1 t}, e^{\rho_2 t}, \ldots, e^{\rho_n t}$ to (A.2.1) are obtained, and the general solution to this homogeneous equation is

$$x(t) = \sum_{k=1}^{n} c_k e^{\rho_k t} \ . \qquad \text{(A.2.3)}$$

However, if one or more of the roots is repeated, then less than n linearly independent solutions are obtained in this way. It may be shown that (see, for example, HILDEBRAND [1, pp. 9-10] or INCE [2,pp. 133-137]) the part of the solution to (A.2.1) corresponding to an m-fold root ρ_1 is of the form

$$\{c_1 + c_2 t + \cdots + c_m t^m\} e^{\rho_1 t} \ .$$

Hence, to each of the n roots of the characteristic equation (A.2.2), repeated roots being counted according to their multiplicity, we can find a corresponding solution to (A.2.1), and the general solution to (A.2.1) is simply a linear combination of these n independent solutions. However, for force-level equations in the LANCHESTER theory of combat,

such repeated roots do not arise unless a force type in our model gets fired upon without returning fire. Unless this happens, such repeated roots do not arise for $F|F$ attrition and its generalizations.

2. Application to LANCHESTER's Equations for Modern Warfare.

The X force-level equation for LANCHESTER's equations of modern warfare (2.2.1) is given by equation (2.2.10) of the main text, which we write here as

$$\frac{d^2x}{dt^2} - ab\,x = 0\,, \tag{A.2.4}$$

whence the characteristic equation is given by

$$\rho^2 - ab = 0\,. \tag{A.2.5}$$

It has two distinct roots $\rho_1 = \sqrt{ab}$ and $\rho_2 = -\sqrt{ab}$ for $ab > 0$ so that the general solution to (A.2.4) is given by

$$x(t) = A'\,e^{\sqrt{ab}\,t} + B'\,e^{-\sqrt{ab}\,t}\,, \tag{A.2.6}$$

where A' and B' are constants, or, in terms of the hyperbolic functions (see Appendix A.1), as

$$x(t) = A\cosh\sqrt{ab}\,t + B\sinh\sqrt{ab}\,t\,, \tag{A.2.7}$$

for equation (2.2.10) of the main text, we can evaluate the constants in (A.2.7) (see Appendix A.1) to obtain

$$x(t) = x_0 \cosh\sqrt{ab}\, t - y_0\sqrt{\frac{a}{b}}\, \sinh\sqrt{ab}\, t \ ,$$

which appears in the main text as equation (2.2.9).

REFERENCES for Appendix A.2

1. F. B. Hildebrand, Advanced Calculus for Engineers, Prentice-Hall, Englewood Cliffs, N.J., 1949.

2. E. L. Ince, Ordinary Differential Equations, Longmans, Green and Co., London, 1927 (reprinted by Dover Publications, Inc., New York, 1956).

APPENDIX A.3: The Generalized RICCATI Equation.

The nonlinear first-order ordinary differential equation

$$\frac{du}{dt} = A(t) + B(t)u + C(t)u^2 , \qquad (A.3.1)$$

is called the generalized RICCATI equation. Although specialized esoteric solution methods have been developed (see BELLMAN [1]), the standard solution method for this nonlinear first-order differential equation is to transform it to a linear second-order differential equation by a transformation of the dependent variable. If we let

$$u = -\frac{1}{C(t)} \frac{w'}{w} , \qquad (A.3.2)$$

where $w' = dw/dt$, then (A.3.1) is transformed into

$$w'' + \{-B - \frac{C'}{C}\} w' + AC\, w = 0 . \qquad (A.3.3)$$

Thus we see the intimate connection between the general linear second-order differential equation and the generalized RICCATI equation.

Although the above connection was already known to LEONHARD EULER (1707-1783) (see WATSON [3, p. 92]), the role played by the RICCATI equation in LANCHESTER combat theory has been recognized only recently. TAYLOR and PARRY [2] have shown that for the differential combat model

$$\frac{dx}{dt} = -a(t)y - \beta(t)x, \quad \text{and} \quad \frac{dy}{dt} = -b(t)x - \alpha(t)y , \qquad (A.3.4)$$

introduction of the force ratio $u = x/y$ yields the following (generalized) RICCATI equation

$$\frac{du}{dt} = b(t)u^2 + \{\alpha(t) - \beta(t)\}u - a(t) \ . \qquad (A.3.5)$$

Finally, as TAYLOR and PARRY [2, p. 525] have emphasized, the primary value of the force-ratio equation (A.3.5) is not for explicitly computing the force ratio, since we have seen that the standard technique for solving the generalized RICCATI equation (A.3.1) is to transform it into a linear second-order equation (A.3.3), which in LANCHESTER-theory applications turns out to be either the X or Y force-level equation. The importance of the force-ratio equation is that it directly provides much useful information about the battle's outcome without one having to spend the time and effort of explicitly computing the force-ratio trajectory (*see*, for example, Section 2.2).

REFERENCES for Appendix A.3

1. R. Bellman, *Methods of Nonlinear Analysis, Volumes 1 and 2*, Academic Press, New York, 1970 (Volume 1) and 1973 (Volume 2).

2. J. Taylor and S. Parry, "Force-Ratio Considerations for Some Lanchester-Type Models of Warfare," *Opns. Res. 23*, 522-533 (1975).

3. G. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944 (Second Edition).

Chapter 3. SOME SIMPLE MODELS OF BATTLE TERMINATION

3.1. Introduction.

As pointed out in Section 2.8, the military operations analyst needs some type of "combat results table" for assessing the outcomes of combat engagements between opposing units in combat models, simulations, and war games. Let us therefore consider how one would construct such a combat results table that relates the engagement's initial conditions to probable outcomes. Recalling from Section 1.3 that there are basically three approaches for assessing the outcomes of tactical engagements (i.e. firepower scores, Monte Carlo simulation, and analytical models), we realize after a little reflection that in all cases there are essentially two aspects of assessing such outcomes: (A1) the dynamics of the engagement, and (A2) the engagement-termination conditions (or "rules").

Although we will proceed analytically via LANCHESTER-type models of warfare, all combat modelling approaches must in some sense include these two aspects. Thus, modelling engagement termination is an essential ingredient for combat analysis, since determination of battle outcome depends on not only the dynamics of combat but also the engagement-termination rules used. Furthermore, any shortcomings in modelling the engagement-termination process are not limited to LANCHESTER-type models: they are basic shortcomings of the state-of-the-art of combat modelling.

It is important for the military operations analyst to have a clear understanding of how force-level and weapon-system-performance factors interact to determine the outcome of battle. In other words, one seeks to answer questions such as, "Who will win the battle? What is the tradeoff between the quality and quantity of weapon systems? When are two forces of equal strength?" For answering such questions, we will assume that the

combat dynamics are given by LANCHESTER-type equations of warfare. With the specification of engagement-termination rules (i.e. an engagement-termination model), we can, of course, determine the outcome of (the simulated) battle simply by plotting the decay of the force levels (or any other state variables) and observing which side reaches its engagement-termination conditions first. Besides being a time consuming approach, this method does not provide any clear understanding of how force-level and weapon-system-performance factors interact to determine the outcome of battle. What is needed are explicit conditions that relate the initial conditions of battle, weapon system capabilities, tactics, and the outcome of battle.

Accordingly, we will develop explicit battle-outcome-prediction conditions for autonomous (i.e. time-invariant) combat dynamics and a certain simple model of engagement termination. We have previously given some of these battle-outcome-prediction results in Section 2.8 without justification. Here we will give theoretical justification for many such results. We must first, however, discuss the modelling of engagement termination. Moreover, the engagement-termination model should be considered to be different and distinct from the combat attrition model.

In this chapter we will consider the modelling of engagement termination and the development of associated battle-outcome-prediction conditions for both deterministic and also stochastic engagement-termination processes. We will consequently be able to answer questions (Q1) through (Q4) posed in Section 2.2 (e.g. "Who will win? What initial force ratio is required to guarantee victory?) for many LANCHESTER-type models. Both theoretical developments and also empirical verification of such models will be discussed.

3.2. Modelling Battle Termination.

As H. K. WEISS [22, has emphasized, engagements that continue until one side is wiped out are rare. Rather, retreat (or disengagement) may begin when the number of casualties sustained by a side approaches 10% or so of its initial strength [22, p. 16]. Possibly the occurrence of some other event (for example, the enemy "taking the high ground") may trigger retreat or disengagement. In any case, though, we should examine the battle-termination process more closely.

Let us therefore consider two forces in ground combat. The engagement begins, the forces interact and casualties are exchanged as battlefield activities are performed, and eventually the battle will end. How did the battle end? Who "won" the battle? What caused the battle to end? These are important (and difficult to answer) questions for the military operations analyst. They are also very important for the combat modeller.

R. L. HELMBOLD [10] has considered that there are four possible outcomes for such a battle:

(O1) one side has been annihilated, with its opponent thereby in undisputed control of the battlefield,

(O2) one side surrenders and submits to the will of its opponent, who thereby gains control of the battlefield,

(O3) neither side surrenders or is annihilated, but one of them has disengaged and either has withdrawn or is in the process of doing so, leaving its opponent in rather clear control of the battlefield.

(O4) neither side has surrendered or been annihilated, but both sides have disengaged and have either withdrawn from the

combat area or are in the process of doing so; the withdrawal is mutual, and control of the battlefield is uncertain for either side.

HELMBOLD [10, pp. 1-2] has considered the battle-termination process further, and he has consequently concluded that outcomes (02) and (03) are the most likely to occur. Thus, he has taken possession of the battlefield as the criterion for victory in battle, although others[1] have stated that additional factors must be considered in evaluating battle outcomes.

However, for simplicity, we will follow HELMBOLD and take possession of the battlefield as the criterion for victory. More precisely, we will determine the winner of an engagement and then assume that he takes possession of the battlefield. Thus, the battle-termination process involves retreat or surrender for the loser. HELMBOLD [10, p. 2] has stated that "in general, a weakening side will prefer to withdraw and abandon the field rather than surrender to its opponent, and (if withdrawal is not feasible) will usually prefer to surrender at some casualty level short of 100 percent total annihilation."

Let us now turn to the modelling of the battle-termination process. Conceptually, we have two forces on the battlefield, each pursuing its own conflicting interests. In general terms, the battle may be considered to be over when one side has decided to abandon its goal (or mission), whatever this may be. In other words, we may say that the battle is over when one unit has ceased to be combat effective. This situation roughly corresponds to HELMBOLD's outcomes (02) and (03) above. In consonance with common military OR usage, let us refer to the onset of the inability of a unit to fulfill its mission as that unit's <u>breakpoint</u>. We will assume that when a unit's breakpoint is reached, the unit will abandon its mission and "break off" the engagement to leave the enemy force is possession of the

field of battle.[2]

The important question for the combat modeller to address is, "What are the significant variables upon which battle termination (i.e. a unit reaching its breakpoint) depends?" Although one can hypothesize many factors upon which battle termination might depend[3] (e.g. casualties, casualty rate, force ratio, tactical situation, perceived tactical situation, etc.), we will assume for simplicity that it depends on the unit's force level[4] as well as the following three factors[5]:

(F1) type of unit,

(F2) size of unit,

and (F3) mission of unit (e.g. attack or defend).

We now formally state these assumptions as the Breakpoint Hypothesis.

BREAKPOINT HYPOTHESIS: A unit will cease to be an effective fighting force in a fire fight when a given force level is reached. When this event happens, the unit loses its ability to perform its mission and will "break off" the engagement. This force-level breakpoint depends on the unit's type, size, and mission.

We will refer to this force level at which a unit ceases to be combat effective as that unit's breakpoint force level (or, simply, its breakpoint). Thus, we are assuming that when a unit's breakpoint is reached, the unit will "break off" the engagement and leave the enemy force in possession of the battlefield. In other words, the first unit that reaches its breakpoint loses the engagement.

Consider now combat between two homogeneous forces (denoted as X and Y) and denote X's breakpoint force level as x_{BP}, with y_{BP} being similarly defined. Then, for example, a Y victory may be described mathematically in the following way:

$$
\text{Y wins when} \begin{cases} \text{(C1)} & x_f = x_{BP}, \\ \text{(C2)} & y_f > y_{BP}, \\ \text{(C3)} & x(t) > x_{BP} \text{ and } y(t) > y_{BP} \text{ for } 0 \leq t < t_f \end{cases}, \tag{3.2.1}
$$

where $x(t)$ and $y(t)$ denote the X and Y force levels at time t, and t_f, $x_f = x(t_f)$, and $y_f = y(t_f)$ denote final values at the end of battle. It is also convenient to write, for example, that

$$
x_{BP} = f_{BP}^{X} x_0 , \tag{3.2.2}
$$

where f_{BP}^{X} denotes a given fraction of X's initial force level. The above Breakpoint Hypothesis implies that f_{BP}^{X} depends on the unit's type, size, and mission. As noted previously in Section 2.8, typical values for a company-sized infantry unit are the following:

$f_{BP}^{X} = 0.7$ for an attacking force,

and

$f_{BP}^{X} = 0.5$ for a defending force.

What happens after a unit reaches its breakpoint? The modelling of subsequent combat actions will, of course, depend on the specific tactical situation being considered. For example, if the Y force cannot disengage (i.e. retreat) upon reaching its breakpoint, then it must either surrender or be annihilated (or at least continue the fire fight at greatly reduced effectiveness to reflect the combat behavior of a fighting unit that has become ineffective and is trying to disengage). If the Y force can retreat, then we might, for example, model combat activities after Y's breakpoint has been reached by a continuous withdrawal of the Y force from battle, a different rate of sustaining casualties for the remaining Y force, and a greatly reduced rate of inflicting casualties for the remaining Y force (to reflect the Y force's lack of combat effectiveness and preoccupation

with retreat).

3.3 Developing Battle-Outcome-Prediction Conditions.

It is important for the military operations analyst to have a clear understanding of how the initial force levels and weapon-system-performance parameters interact to determine the outcome of battle. For any particular battle (e.g. in the stationary case, for specified values of the attrition-rate coefficients and initial force levels), we can always, of course, determine the outcome by explicitly computing the force-level trajectories and plotting their decay over time: the loser is simply the side that first reaches its breakpoint. This approach, however, is time consuming and by itself tells us essentially nothing about the parametric dependence of battle outcome on initial force levels and weapon-system-performance parameters.

It is therefore of interest to develop victory-prediction conditions, which facilitate sensitivity analysis and help one obtain insights into the dynamics of combat by explicitly portraying the relationship between these various factors in the combat-attrition process and battle outcome. As we have discussed just above and in Section 2.8, such battle-outcome-prediction conditions depend not only on the combat dynamics (e.g. LANCHESTER-type differential equations) but also on the battle-termination model. We will assume here that the battle ends when one side first reaches its breakpoint force level (see the Breakpoint Hypothesis of Section 3.2). We will then see that for certain battle dynamics we need not spend the time and effort of explicitly computing force-level trajectories in order to determine the victor in such fixed-force-level-breakpoint battles. We have already given in Section 2.8 special cases of such victory-prediction conditions for LANCHESTER's classic combat formulations. Furthermore, the force-annihilation-prediction conditions that we developed in Sections 2.2 and 2.4 (see, for

example, Proposition 2.2.1) are also special cases of these more general victory-prediction conditions.

We will now present two different methods for developing conditions that predict the outcome of fixed-force-level-breakpoint battles between two homogeneous forces with fairly general combat dynamics. These two methods for developing battle-outcome-prediction conditions involve

(A) determining the minimum of two first-passage times,

and (B) using the time-independent coupling of the force levels.

Both approaches many times lead to conditions that predict the outcome of battle without having to spend the time and effort of explicitly computing force-level trajectories. Restrictions that must be placed on the combat dynamics for each of these two approaches are briefly discussed. Applications of these methods to specific LANCHESTER-type combat models (generally LANCHESTER's classic combat formulations) are given in Sections 3.6 through 3.9 below.

The first (and conceptually more general) way to develop victory-prediction conditions for a fixed-force-level-breakpoint battle with general battle dynamics is to determine the minimum of the first passage times for each side's force level going through its breakpoint. We will refer to this approach as <u>Method A</u>. Let us denote the first-passage time for X's force level going through its breakpoint force level x_{BP} as t^X_{BP}, and similarly for t^Y_{BP}. We have then, for example, from the definition of t^X_{BP} that $x(t) > x_{BP}$ for all $t \in [0, t^X_{BP})$. It follows that t^X_{BP} is the smallest positive root of the equation

$$x(t^X_{BP}) = x_{BP} = f^X_{BP}\, x_0 \ . \qquad (3.3.1)$$

in calculating $x(t)$ for the determination of t_{BP}^{X}, we will assume that the two forces never disengage, i.e. the combat dynamics hold for all time. We will also set $t_{BP}^{X} = +\infty$ if no such positive root to equation (3.3.1) exists.

It follows then that, for example, Y will win if and only if $t_{BP}^{X} < t_{BP}^{Y}$. This situation is shown in Figure 3.1, in which, for example, we have terminated the force-attrition process for Y once his breakpoint has been reached (i.e. $y(t) = y_{BP}$ for all $t \geq t_{BP}^{Y}$). In actuality (if we assume that disengagement is possible), however, attrition for both sides stops at $t_{W}^{Y} = t_{BP}^{X} < t_{BP}^{Y}$, as shown in Figure 3.2. Here t_{W}^{Y} denotes the time for Y to win a fixed-force-level-breakpoint battle. Although Method A conceptually applies to any LANCHESTER-type attrition process, victory-prediction-condition results have so far only been obtained for LANCHESTER's classic combat formulations (i.e. the F|F and FT|FT attribition processes) by this method.

The second (and conceptually more restrictive) way[6] to develop victory-prediction conditions for a fixed-force-level-breakpoint battle between two homogeneous forces involves use of HELMBOLD's monotonicity condition that one force level must be a strictly increasing function of the other one, i.e.

$$x = g(y) , \tag{3.3.2}$$

where $g(y)$ is strictly increasing for $y_f \leq y \leq y_0$. The desired victory-prediction conditions readily follow from such a monotonicity condition. We will refer to this approach as <u>Method B</u>. The monotonicity condition is developed, however, from the state equation (<u>see</u> Section 2.2) so that this approach is limited to LANCHESTER-type models for which a state

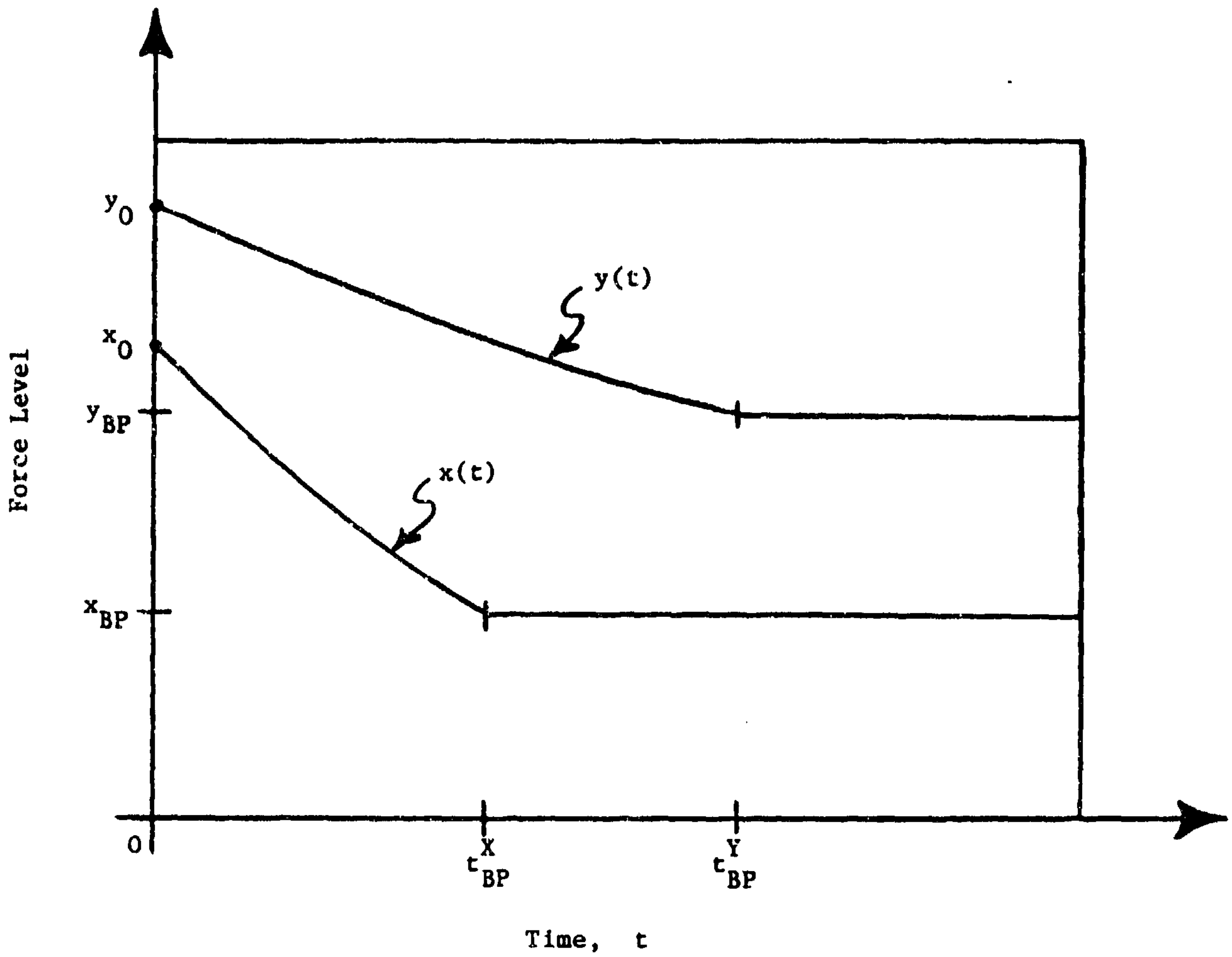


Figure 3.1. Relationship between t_{BP}^X and t_{BP}^Y for a Y victory.

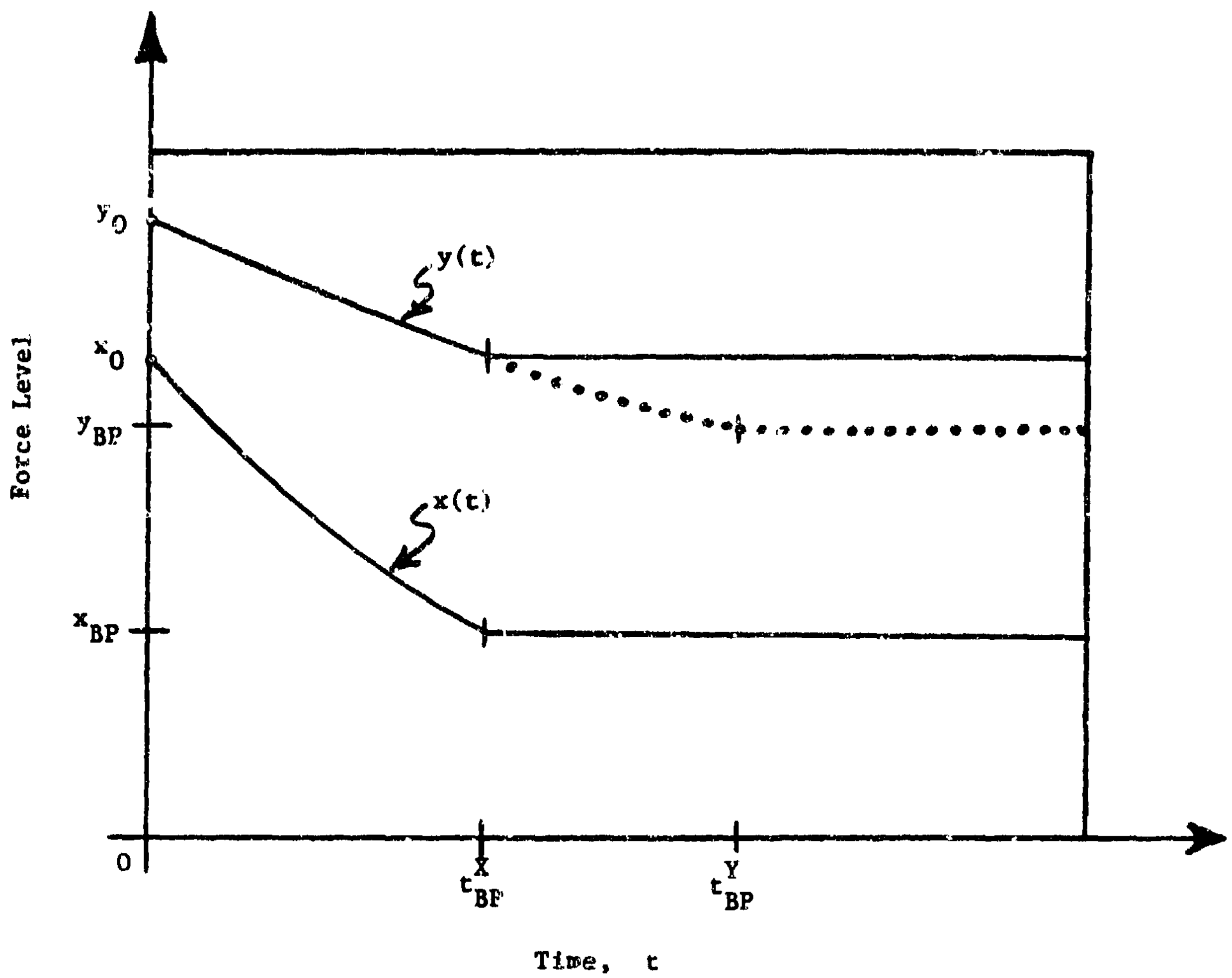


Figure 3.2. Decay of force levels for $0 \le t \le t^Y_W = t^X_{BP} < t^Y_{BP}$ in the case of a Y victory. The dotted line shows what the decay of the Y force level would be if the forces did not disengage at $t = t^X_{BP}$.

equation of the form (2.2.3) holds.[7] It will be shown below that (provided a certain additional "reasonable" technical condition is satisfied) Y will win (in finite time) such a battle if and only if

$$x_{BP} > g(y_{BP}) \quad . \tag{3.3.3}$$

This is the key result for developing victory-prediction conditions by Method B, since $g = g(y;x_0,y_0)$.

Let us now demonstrate the validity of the fairly general victory-prediction condition, i.e. (3.3.3), indicated in the preceding paragraph. First, let us restate (3.3.2) somewhat more formally as Condition (FI).

> CONDITION (FI): The X and Y force levels are (deterministically) related to each other by $x = g(y) = g(y;x_0,y_0)$, where $g(y)$ is strictly increasing for $y_f \leq y \leq y_0$ and y_f denotes the final Y force level at the end of battle.

Then, the key result for battle-outcome prediction by Method B is (3.3.3), which we restate as Proposition 3.3.1.

> PROPOSITION 3.3.1: Assume that Condition (FI) holds and that t^X_{BP} is finite. Then, Y will win a fixed-force-level-breakpoint battle in finite time if and only if $x_{BP} > g(y_{BP}) = g(y_{BP};x_0,y_0)$.

PROOF: To prove necessity, we assume that Y wins, which implies that $x_f = x_{BP}$ and $y_f > y_{BP}$. It follows that $g(y_f) > g(y_{BP})$, since $g(y)$ is strictly increasing. Thus, $x_{BP} = x_f = g(y_f) > g(y_{BP})$, so that $x_{BP} > g(y_{BP})$, and necessity has been proved. To prove sufficiency, we assume that $x_{BP} > g(y_{BP})$. It follows that $g[y(t)] = x(t) > x_{BP} > g(y_{BP})$, whence $y(t) > y_{BP}$ for $0 \leq t \leq t_f$, since $g(y)$ is strictly increasing. Also, t^X_{BP} being finite implies that $x(t) > x_{BP}$ for $0 \leq t < t_f$ but $x_f = x(t_f) = x(t^X_{BP}) = x_{BP}$ with t^X_{BP} finite. Hence, Y will win the battle in finite time.[8] Q.E.D.

As we have just seen, this second method of developing victory-prediction conditions depends in an essential way on Condition (FI). So far we have not discussed which class of LANCHESTER-type equations (if any) corresponds to (3.3.2). We will now show that a certain class of rather simple LANCHESTER-type equations for combat between two homogeneous forces (although somewhat restrictive) does indeed yield (3.3.2). Furthermore, the simple combat models (2.2.1) and (2.4.1) belong to this class of LANCHESTER-type equations.

Thus, we consider the following LANCHESTER-type equations

$$\begin{cases} \dfrac{dx}{dt} = -f(t)\, F_1(x)\, F_2(y) & \text{with } x(0) = x_0, \\[2ex] \dfrac{dy}{dt} = -f(t)\, G_1(x)\, G_2(y) & \text{with } y(0) = y_0, \end{cases} \tag{3.3.4}$$

where $f(t) > 0$ almost everywhere (e.g. except for a finite number of points) in time, and F_1, F_2, G_1, and $G_2 > 0$ for $x > x_{BP}$ and $y > y_{BP}$. It follows that

$$p(x_0) - p(x) = q(y_0) - q(y) \text{ ,} \tag{3.3.5}$$

where

$$p(x_0) - p(x) = H(x_0,x) = \int_x^{x_0} \frac{G_1(\xi)}{F_1(\xi)}\, d\xi, \tag{3.3.6}$$

and

$$q(y_0) - q(y) = K(y_0,y) = \int_y^{y_0} \frac{F_2(\xi)}{G_2(\xi)}\, d\xi \ . \tag{3.3.7}$$

It is readily seen that $p(x)$ is strictly increasing for $x_{BP} \leq x \leq x_0$; similarly, $q(y)$ is strictly increasing for $y_{BP} \leq y \leq y_0$. Hence, the inverse function $p^{-1}(\eta)$ is also strictly increasing for $p(x_{BP}) \leq \eta \leq p(x_0)$. It therefore follows from (3.3.4) that

$$x = p^{-1}[p(x_0) - q(y_0) + q(y)] \ , \tag{3.3.8}$$

for $y_{BP} \leq y_f \leq y \leq y_0$. Thus, for the battle dynamics (3.3.4), we can always develop a functional relationship between the force levels of the form of (3.3.2). In other words, we have proven the following proposition

> PROPOSITION 3.3.2: Condition (FI) is satisfied for all LANCHESTER-type equations with two force-level variables of the form (3.3.4).

HELMBOLD [10], however, has developed his pioneering results in a different form: he has given his victory-prediction conditions (somewhat less explicitly) in terms of the casualty fractions of the combatants. We will now show how our results are equivalent to his. Let us accordingly denote X's casualty fraction as f_c^X, i.e. $f_c^X = (x_0-x)/x_0$, and similarly

for f_c^Y. We will also denote X's breakpoint casualty fraction corresponding to x_{BP} as $(f_c^X)_{BP}$, and similarly for $(f_c^Y)_{BP}$. Then corresponding to Condition (FI) we have Condition (CFI).

CONDITION (CFI): The X and Y casualty fractions are (deterministically) related to each other by $f_c^X = \varphi(f_c^Y) = \varphi(f_c^Y;x_0,y_0)$ where $\varphi(f_c^Y)$ is strictly increasing for $0 \leq f_c^Y \leq (f_c^Y)_f$ and $(f_c^Y)_f$ denotes the final Y casualty fraction at the end of battle.

It is easy to show that Conditions (FI) and (CFI) are equivalent.

PROPOSITION 3.3.3: $x = g(y)$ with $g(y)$ strictly increasing if and only if $f_c^X = \varphi(f_c^Y)$ with $\varphi(f_c^Y)$ strictly increasing.

Consequently, the following is the analogue of Proposition 3.3.1.

PROPOSITION 3.3.1': Assume that Condition (CFI) holds and that t_{BP}^X is finite. Then Y will win a fixed-force-level-breakpoint battle in finite time if and only if $(f_c^X)_{BP} < \varphi[(f_c^Y)_{BP}] = \varphi[(f_c^Y)_{BP};\ x_0,\ y_0]$.

It is very convenient when, for example, the initial force level x_0 and the casualty fraction f_c^X are "separable" in $H(x_0,x)$, i.e.

$$H(x_0,x) = \gamma(x_0)\ h(f_c^X)\ . \tag{3.3.9}$$

Let us therefore introduce the concept of a function being quasi-homogeneous: we will refer to a function of two variables $F(x,y)$ as being quasi-homogeneous when

$$F(x,y) = \lambda(x)\ f(\frac{y}{x}) \ .$$

Hence, if $H(x_0,x)$ and $K(y_0,y)$ are quasi-homogeneous functions, then

$$H(x_0,x) = \gamma(x_0)\ h_1(\frac{x}{x_0}) = \gamma(x_0)\ h(f_c^X) \ , \tag{3.3.10}$$

and

$$K(y_0,y) = \lambda(y_0)\ k_1(\frac{y}{y_0}) = \lambda(y_0)\ k(f_c^Y) \ , \tag{3.3.11}$$

where $\gamma(\xi)$, $\lambda(\xi) > 0$ for $\xi > 0$. It is easily shown that $h(f_c^X)$ is strictly increasing and positive for $0 < f_c^X \le (f_c^X)_{BP}$; similarly, $k(f_c^Y)$ is strictly increasing for $0 < f_c^Y \le (f_c^Y)_{BP}$. It follows that

$$f_c^X = h^{-1}\left[\frac{\lambda(y_0)}{\gamma(x_0)}\ k(f_c^Y)\right] = \varphi(f_c^Y) \ , \tag{3.3.12}$$

where h^{-1} denotes the inverse function and $\varphi(f_c^Y)$ is strictly increasing for $0 \le f_c^Y \le (f_c^Y)_{BP}$. Also,

$$x = x_0 h_1^{-1}\left[\frac{\lambda(y_0)}{\gamma(x_0)}\ k_1(\frac{y}{y_0})\right] = g(y) \ , \tag{3.3.13}$$

where h_1^{-1} denotes the inverse function, and $g(y)$ is strictly increasing for $y_{BP} \le y \le y_0$.

As we have noted above, we actually have that

$$\varphi = \varphi(f_c^Y; x_0, y_0) \ . \tag{3.3.14}$$

It is of interest in military OR to consider the case in which results do not depend on the absolute initial force levels but on the initial force ratio, i.e.

$$\varphi(f_c^Y; x_0, y_0) = \psi(f_c^Y; x_0/y_0) \quad . \tag{3.3.15}$$

In this case, (3.3.12) implies that there is a $\phi = \phi(x_0/y_0)$ such that

$$\phi(x_0/y_0) = \frac{\gamma(x_0)}{\lambda(y_0)} \quad . \tag{3.3.16}$$

However, (3.3.16) is equivalent to the functional equation

$$f(xy) = g(x)\ h(y) \quad ,$$

which has the general solution [1, pp. 144-145]

$$g(t) = at^c, \qquad h(t) = bt^c \quad ,$$

so that the only functions that satisfy (3.3.16) are

$$\gamma(x_0) = C_1 x_0^c \quad , \qquad \lambda(y_0) = C_2 y_0^c \quad . \tag{3.3.17}$$

Thus, "absorbing the constant C_1 into the function h_1," we may write when $H(x_0,x)$ and $K(y_0,y)$ are quasi-homogeneous functions and (3.3.15) holds

$$H(x_0,x) = x_0^c h_1\left(\frac{x}{x_0}\right) \quad , \tag{3.3.18}$$

so that both $H(x_0,x)$ and $K(y_0,y)$ are homogeneous functions of degree c when they are quasi-homogeneous and results do not depend on the absolute initial force levels but only on the initial force ratio.

Now let us show how Proposition 3.3.1 and (3.3.13) yield an explicit victory-prediction condition when $H(x_0,x)$ and $K(y_0,y)$ are quasi-homogeneous. We begin by observing that Proposition 3.3.1 and (3.3.13) yields

$$\frac{x_{BP}}{x_0} = f_{BP}^X > h_1^{-1}\left[\frac{\lambda(y_0)}{\gamma(x_0)}\, k_1(f_{BP}^Y)\right] . \tag{3.3.19}$$

Consider next the quantity $\gamma(x_0)\, h_1(x_{BP}/x_0)$, where $h_1(\xi)$ is strictly increasing and positive for $0 \leq \xi < 1$. Making use of (3.3.19), we find that

$$\gamma(x_0)\, h_1(f_{BP}^X) < \gamma(x_0)\, h_1\left(h_1^{-1}\left[\frac{\lambda(y_0)}{\gamma(x_0)}\, k_1(f_{BP}^Y)\right]\right) ,$$

so that when t_{BP}^X is finite and $H(x_0,x)$ and $K(y_0,y)$ are quasi-homogeneous,

$$Y \text{ will win if and only if } \quad \frac{\gamma(x_0)}{\lambda(y_0)} < \frac{k_1(f_{BP}^Y)}{h_1(f_{BP}^X)} . \tag{3.3.20}$$

Furthermore, the length of battle is finite. We will not repeat this fact any more, although it does hold for all the victory-prediction conditions in the remainder of this section, since we will always assume that t_{BP} is finite. Also, let us continue to assume that $H(x_0,x)$ and $K(y_0,y)$ are quasi-homogeneous. Then, we may also show that

$$Y \text{ will win if and only if } \quad \frac{\gamma(x_0)}{\lambda(y_0)} < \frac{k[(f_c^Y)_{BP}]}{h[(f_c^X)_{BP}]} . \tag{3.3.21}$$

Recalling (3.3.18), we also see that when $\gamma(x_0)/\lambda(y_0) = \phi(x_0/y_0)$ and $c > 0$, the above victory-prediction conditions become

$$Y \text{ will win if and only if } \quad \frac{x_0}{y_0} < \sqrt[c]{\frac{k_1(f_{BP}^{Y})}{h_1(f_{BP}^{X})}} \,, \tag{3.3.22}$$

and

$$Y \text{ will win if and only if } \quad \frac{x_0}{y_0} < \sqrt[c]{\frac{k[(f_c^{Y})_{BP}]}{h[(f_c^{X})_{BP}]}} \,. \tag{3.3.23}$$

In other words, when the functions $H(x_0,x)$ and $K(y_0,y)$ are quasi-homogeneous, we have the explicit victory-prediction condition (3.3.20) or, equivalently, (3.3.21). If additionally $\gamma(x_0)/\lambda(y_0) = \phi(x_0/y_0)$ and $c > 0$, these results simplify still further. We will see below that the victory-prediction conditions given in Section 2.8 (namely, (2.8.3) and (2.8.13)) are special cases of (3.3.22). For a discussion of the insights into the dynamics of combat to be gained from such victory-prediction conditions, the reader is directed to Section 2.8.

3.4. Modelling a Unit's Force-Level Breakpoint.

Thus, an engagement's outcome depends on both the combat dynamics and also the battle-termination model. Even having decided to assume the Breakpoint Hypothesis (see Section 3.2 above) for combat between two homogeneous forces (denoted as X and Y), we are not finished with battle-termination modelling: for a force-level-breakpoint battle, we can model a unit's breakpoint as being either a random variable (realized before the battle) or a deterministic quantity. Thus, we have the two general models for a side's battle-termination process:

(M1) deterministic breakpoint,

(M2) random breakpoint.[9]

Clearly, a deterministic breakpoint may be considered to be a special case of a random breakpoint. The latter is used to model battle termination via a so-called "break curve," which gives the probability that a force will discontinue the engagement as a function of its current force level[10] (usually normalized as a fraction of the initial force level).

Figure 3.3 shows a hypothetical force-level break curve for the X force. We may think of such a break curve as modelling battle termination in the following manner. At or before the beginning of battle, a sample breakpoint force level is drawn from the distribution of such values as given by the appropriate break curve. This is done for each side, and the values so drawn are called the "breakpoints" of the two sides. The battle then begins and continues until one side's force level becomes equal to its preselected breakpoint. At this point, the side whose preselected breakpoint has been reached is said to "break," meaning that it is presumed to abandon its mission and to discontinue or "break off" the engagement. Thus, the side that breaks is the loser according to this model.

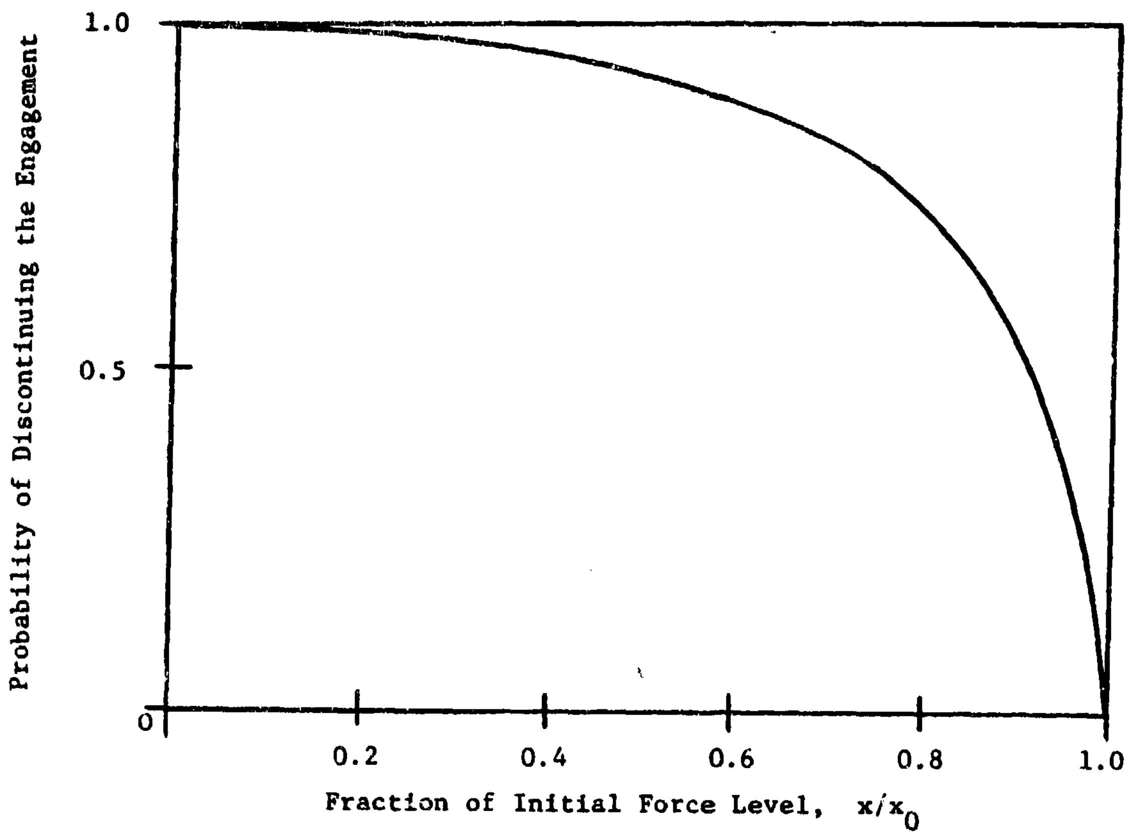


Figure 3.3. Hypothetical force-level break curve for the X force.

A deterministic break curve may be thought to be a special case of such a random break curve. A deterministic force-level break curve is shown in Figure 3.4. For such a deterministic break curve, we can immediately apply the results of Section 3.3 (e.g. Proposition 3.3.1 or (3.3.20)) and obtain battle-outcome-prediction conditions. However, if the breakpoints are random variables, then further analysis is required as shown in Section 3.7.

Before continuing further, let us point out that even if we consider that the force levels are the significant variables in the battle-termination process and that the breakpoints are either deterministic or stochastic, we still may consider two different types of battle-termination models:

(T1) descriptive,

(T2) adaptive behavioral.

By a descriptive model of the battle-termination process, we mean a model that describes the battle-termination process in terms of one or more independent variables, like the models described above. Consequently, such a descriptive model can give us (if only in a probabilistic sense) each side's breakpoint before the battle begins.

By an adaptive behavioral model, we mean one in which each side considers the progress of the battle and accordingly decides whether or not to continue the engagement. In such a model, each side behaves according to the results of a dynamic rational-decision process rather than simply preselecting a specific breakpoint. HELMBOLD [10] has shown, however, that both models are equivalent in the simple case in which each side governs its behavior according to only its own state (i.e. own casualty fraction). He has concluded [10, p. 5] that break curves (i.e. a descriptive model of battle termination) do reflect the dynamic decision process taking place in combat unless, for example, one side's breakpoint distribution depends on

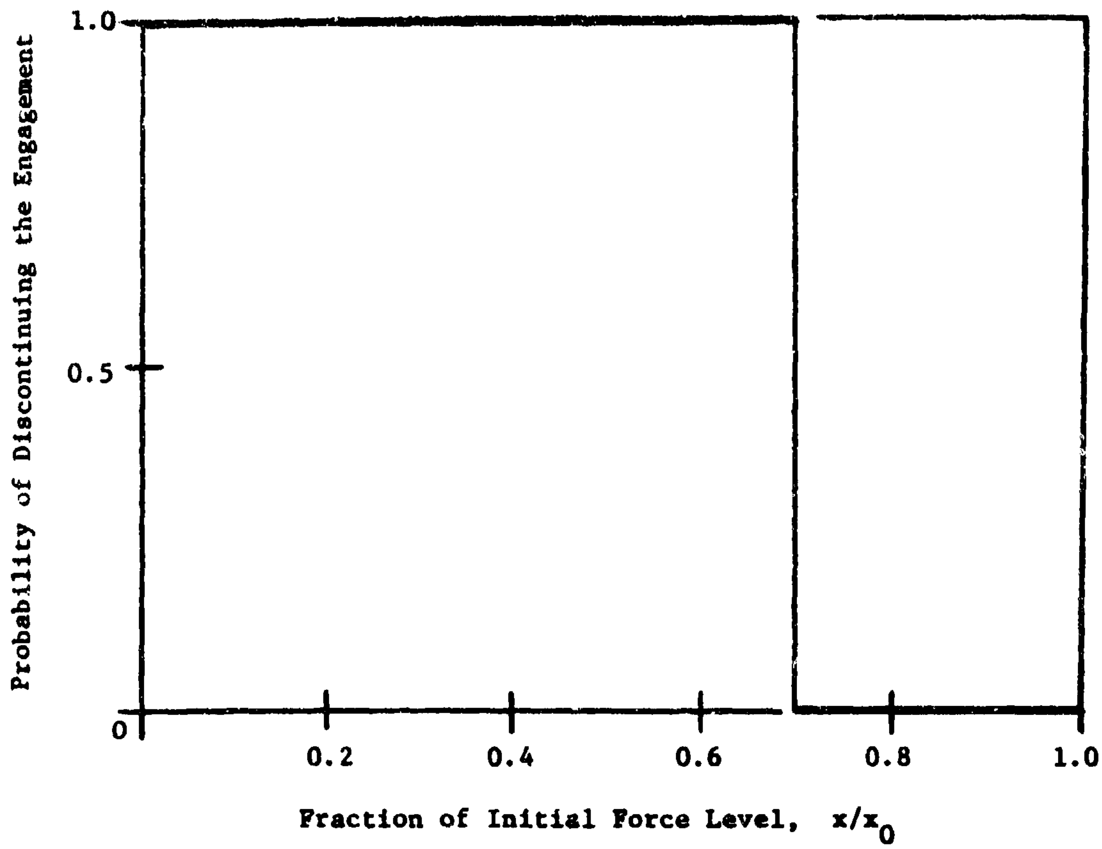


Figure 3.4. A deterministic force-level break curve.

the other side's casualty level.

A final word of caution to the reader. Unfortunately, HELMBOLD [10] has shown (see also Section 3.13 below) that if one considers two homogeneous forces in deterministic LANCHESTER -type combat without replacements and withdrawals and assumes

(A1) a breakpoint hypothesis is applicable to all battles,[11]

and (A2) a universally applicable deterministic attrition process,

then the breakpoint hypothesis "yields theoretical implications that are at variance with the available battle-termination data in several essential aspects." HELMBOLD [10] has considered random breakpoints in his work. Thus, such a simple model of battle termination is not scientifically valid. Nevertheless, this simple battle-termination model is widely used in defense analyses (see HELMBOLD [10] for numerous examples), and the author knows of no alternative battle-termination model that has been widely used and has passed the same stringent scientific test of validity that this simple model has failed. Thus, we will continue to assume that our Breakpoint Hypothesis is true. It therefore seems appropriate to develop the relationship between the initial force levels, weapon-system-performance factors, and the outcome of battle for simple LANCHESTER-type models such as (2.2.1) and (2.4.1) above.

3.5. Victory-Prediction Conditions for Deterministic LANCHESTER-Type Attrition Processes with Deterministic Force-Level Breakpoints.

When both sides' force-level breakpoints are deterministic, then Method B of Section 3.3 provides some explicit victory-prediction conditions (namely, Propositions 3.3.1 and 3.3.1'). For convenience, we restate them here as Propositions 3.5.1 and 3.5.1':

PROPOSITION 3.5.1: Assume that Condition (FI) holds, that t_{BP}^{X} is finite, and that the breakpoints are deterministic. Then, Y will win a fixed-force-level-breakpoint battle in finite time if and only if $x_{BP} > g(y_{BP}) = g(y_{BP};x_0,y_0)$.

PROPOSITION 3.5.1': Assume that Condition (CFI) holds, that t_{BP}^{X} is finite, and that the breakpoints are deterministic. Then, Y will win a fixed-force-level-breakpoint battle in finite time if and only if $(f_c^X)_{BP} < \varphi[(f^Y)_{BP}] = \varphi[(f_c^Y)_{BP};x_0,y_0]$.

Moreoever, we know by Proposition 3.3.2 that Condition (FI) (equivalently, Condition (CFI)) holds for all LANCHESTER-type equations of the form

$$\begin{cases} \dfrac{dx}{dt} = -f(t)\ F_1(x)\ F_2(y) & \text{with } x(0) = x_0\ , \\ \\ \dfrac{dy}{dt} = -f(t)\ G_1(x)\ G_2(y) & \text{with } y(0) = y_0\ , \end{cases} \tag{3.5.1}$$

where $f(t) > 0$ almost everywhere (e.g. except for a finite number of points) in time, and F_1, F_2, G_1, and $G_2 > 0$ for $x > x_{BP}$ and $y > y_{BP}$.

Furthermore, for combat modelled by the LANCHESTER-type equations (3.5.1), we showed in Section 3.3 that when $H(x_0,x)$ and $K(y_0,y)$ [defined by (3.3.6) and (3.3.7), respectively] are quasi-homogeneous functions and t_{BP}^X is finite, then

$$\text{Y will win in finite time if and only if} \quad \frac{\gamma(x_0)}{\lambda(y_0)} < \frac{k_1(f_{BP}^Y)}{h_1(f_{BP}^X)}, \qquad (3.5.2)$$

where h_1 and k_1 are defined by (3.3.10) and (3.3.11) and, for example, $x_{BP} = f_{BP}^X\, x_0$, or (equivalently)

$$\text{Y will win in finite time if and only if} \quad \frac{\gamma(x_0)}{\lambda(y_0)} < \frac{k[(f_c^Y)_{BP}]}{h[(f_c^X)_{BP}]}, \qquad (3.5.3)$$

where h and k are also defined by (3.3.10) and (3.3.11) and, for example, $(f_c^X)_{BP} = (x_C - x_{BP})/x_0$. Additionally, when $\gamma(x_0)/\lambda(y_0) = \phi(x_0/y_0)$ and $c > 0$, the above victory-prediction conditions simplify to (<u>see</u> Section 3.3 for proof)

$$\text{Y will win in finite time if and only if} \quad \frac{x_0}{y_0} < \sqrt[c]{\frac{k_1(f_{BP}^Y)}{h_1(f_{BP}^X)}}, \qquad (3.5.4)$$

and (equivalently)

$$\text{Y will win in finite time if and only if} \quad \frac{x_0}{y_0} < \sqrt[c]{\frac{k[(f_c^Y)_{BP}]}{h[(f_c^X)_{BP}]}}. \qquad (3.5.5)$$

The above are fairly general victory-prediction conditions. In the next section we apply them to LANCHESTER's classic combat formulations (2.2.1) and (2.4.1) (i.e. LANCHESTER-type equations for the $F|F$ attrition process and also for the $FT|FT$ process, respectively). A discussion of the insights into the dynamics of combat to be gained from such victory-prediction conditions is to be found in Section 2.8.

3.6. Development of Victory-Prediction Conditions for LANCHESTER's Classic Models with Deterministic Force-Level Breakpoints.

In this section we will show how to develop victory-prediction conditions by the two methods discussed above in Section 3.3 (i.e. Methods A and B) for LANCHESTER's two classic combat models (2.2.1) and (2.4.1), i.e. the equations for the $F|F$ attrition process and for the $FT|FT$ process. We have previously given these results without justification in Section 2.8, where a detailed examination and discussion of the insights into the dynamics of combat to be gained from these victory-prediction conditions is, however, given. For both these two combat models, we have a choice of which method to use for developing victory-prediction conditions for deterministic breakpoints. Method B, however, is better suited to combat modelled with random breakpoints and is therefore very important for our developments in the next couple of sections.

We first consider that the combat dynamics are given by LANCHESTER's equations for modern warfare (2.2.1). First, we will develop a victory-prediction condition by Method A, i.e. by determining the minimum of the first passage times for each side's force level to reach its breakpoint value. For (2.2.1) the time for the X force to reach its breakpoint, denoted as t_{BP}^{X}, may be obtained by determining the smallest positive root of the equation (3.3.1) with $x(t)$ given by (2.2.8). Consequently, we find that

$$t_{BP}^{X} = \begin{cases} \dfrac{1}{\sqrt{ab}} \ln\left\{ \dfrac{x_0}{x_{BP}} \right\} & \text{for } \dfrac{x_0}{y_0} = \sqrt{\dfrac{a}{b}}\,, \\[2ex] \dfrac{1}{\sqrt{ab}} \ln\left\{ \dfrac{-x_{BP} + \sqrt{\frac{a}{b} y_0^2 - x_0^2 + x_{BP}^2}}{y_0 \sqrt{\frac{a}{b}} - x_0} \right\} & \text{for } \dfrac{x_0}{y_0} \neq \sqrt{\dfrac{a}{b}} \end{cases} \tag{3.6.1}$$

and similarly for t_{BP}^{Y}. When t_{BP}^{X} is not defined, we will take it to be $(+\infty)$. It follows [with three cases having to be considered: (A) $x_0/y_0 < \sqrt{a/b}$, (B) $x_0/y_0 = \sqrt{a/b}$, and (C) $x_0/y_0 > \sqrt{a/b}$] that

$$t_{BP}^{X} < t_{BP}^{Y} \text{ if and only if } a(y_0^2 - y_{BF}^2) > b(x_0^2 - x_{BP}^2) . \qquad (3.6.2)$$

Letting $x_{BP} = f_{BP}^{X} x_0$ and $y_{BP} = f_{BP}^{Y} y_0$, we may state the above result (3.6.2) as

PROPOSITION 3.6.1: When the combat dynamics are given by LANCHESTER's equations for modern warfare (2.2.1), Y will win a fixed-force-level-breakpoint battle in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}\left\{\frac{1-(f_{BP}^{Y})^2}{1-(f_{BP}^{X})^2}\right\}} . \qquad (3.6.3)$$

The victory-prediction condition (3.6.3) of Proposition 3.6.1 was given previously without justification in Section 2.8 as (2.8.3).

Alternatively, we could have developed Proposition 3.6.1 by Method B, i.e. by using HELMBOLD's monotonicity condition that one force level must be a strictly increasing function of the other one. By Proposition 3.3.2, we know that Condition (FI) is satisfied for the model (2.2.1) so that we could use Proposition 3.5.1 (equivalently, Proposition 3.3.1) to prove Proposition 3.6.1. However, it is much more convenient to invoke the victory-prediction condition (3.5.4), which is a special case of the victory-prediction condition in Proposition 3.5.1 (namely,

(3.3.3)). To invoke (3.5.4), we observe that $H(x_0,x)$ (as given by (3.3.6)) is quasi-homogeneous and (3.3.10) yields that

$$\gamma(x_0) = x_0^2, \quad \text{and} \quad h_1\left(\frac{x}{x_0}\right) = b\left\{1 - \left(\frac{x}{x_0}\right)^2\right\}, \tag{3.6.4}$$

since $H(x_0,x) = b(x_0^2 - x^2)$. Similarly,

$$\lambda(y_0) = y_0^2, \quad \text{and} \quad k_1\left(\frac{y}{y_0}\right) = a\left\{1 - \left(\frac{y}{y_0}\right)^2\right\}. \tag{3.6.5}$$

Consequently, $\lambda(x_0)/\gamma(y_0) = \phi(x_0/y_0) = (x_0/y_0)^2$ so that $c = 2 > 0$, and we may invoke (3.5.4) to prove Proposition 3.6.1 provided that t_{BP}^X is finite. Thus, we must again consider t_{BP}^X as given by (3.6.1). Proposition 3.6.1, however, requires that

$$\frac{a}{b}\, y_0^2 \geq \frac{a}{b}\,(y_0^2 - y_{BP}^2) > x_0^2 - x_{BP}^2, \tag{3.6.6}$$

so that (3.6.1) yields that t_{BP}^X is well defined and finite. We finally note that the force-annihilation-prediction condition given in Proposition 2.2.1 may be obtained from Proposition 3.6.1 by setting $f_{BP}^X = f_{BP}^Y = 0$.

Let us now give an example that shows that the requirement that t_{BP}^X be finite is absolutely necessary for Proposition 3.5.1 to be true. Accordingly, we consider the following variable-coefficient LANCHESTER-type equations[12]

$$\begin{cases} \dfrac{dx}{dt} = -a(t)y & \text{with} \quad x(0) = x_0, \\[2ex] \dfrac{dy}{dt} = -b(t)x & \text{with} \quad y(0) = y_0, \end{cases} \tag{3.6.7}$$

where

$$a(t) = k_a h(t), \qquad b(t) = k_b h(t) , \tag{3.6.8}$$

$h(t) > 0$ for all $t \geq 0$, and k_a and k_b are constants. We observe that the equations (3.6.7) are a special case of the equations (3.3.4) so that we know by Proposition 3.3.2 that Condition (FI) holds for them. In fact, we have the following square law[13]

$$k_b(x_0^2 - x^2) = k_a(y_0^2 - y^2) . \tag{3.6.9}$$

Moreover, the substitution

$$s = \sqrt{k_a k_b} \int_0^t h(u)\, du , \tag{3.6.10}$$

transforms (3.6.7) into

$$\begin{cases} \dfrac{dx}{ds} = -\sqrt{\lambda_R}\, y & \text{with } x(0) = x_0 , \\ \dfrac{dy}{ds} = -\dfrac{1}{\sqrt{\lambda_R}}\, x & \text{with } y(0) = y_0 , \end{cases} \tag{3.6.11}$$

where the relative-fire-effectiveness parameter λ_R is defined by

$$\lambda_R = k_a/k_b . \tag{3.6.12}$$

Consequently, $x(s)$ is given by [14]

$$x(s) = x_0 \cosh s - y_0\sqrt{\lambda_R} \sinh s ,$$

whence

$$x(t) = x_0 \cosh \theta(t) - y_0\sqrt{\frac{k_a}{k_b}} \sinh \theta(t) , \tag{3.6.13}$$

where

$$\theta(t) = \sqrt{k_a k_b} \int_0^t h(s)\, ds .$$

It now follows, however, that

$$\frac{x_0}{y_0} < \sqrt{\frac{k_a}{k_b} \frac{\{1 - (f_{BP}^Y)^2\}}{\{1 - (f_{BP}^X)^2\}}} \tag{3.6.14}$$

does not always imply that the X force will lose such a fixed-force-level-breakpoint battle with combat dynamics (3.6.7) (as Proposition 3.5.1 implies it should when t_{BP}^X is finite).

The nonsufficiency of (3.6.14) to effect a Y victory occurs when t_{BP}^X is not finite, i.e. when $\lim_{t \to +\infty} \theta(t) = M < +\infty$. For example, consider a fire fight in which the combatants take cover and continue to reduce their vulnerability so that each side's fire effectiveness decays exponentially over time, i.e. $a(t) = k_a e^{-\gamma t}$ and $b(t) = k_b e^{-\gamma t}$. Then

$$\theta(t) = \frac{\sqrt{k_a k_b}}{\gamma} (1 - e^{-\gamma t}) ,$$

and $\lim_{t \to +\infty} \theta(t) = M = \sqrt{k_a k_b}/\gamma$. Hence, when (3.6.14) holds, we can always choose γ so that $\lim_{t \to +\infty} x(t) = x_0 \cosh M - y_0\sqrt{\lambda_R} \sinh M > x_{BP}$. In other words, we can always pick γ so that t_{BP}^X is not finite, and then (3.6.14) does not imply that the X force will lose such a fixed-force-level-breakpoint battle.

We next consider that the combat dynamics are given by constant-coefficient LANCHESTER-type equations for an FT|FT attrition process (2.4.1). We will again first develop victory-prediction conditions by Method A. For (2.4.1) the time for the X force to reach its breakpoint may be obtained by determining the smallest positive root of the equation (3.3.1) with $x(t)$ given by (2.4.10). Accordingly, we find that

$$t_{BP}^{X} = \begin{cases} \frac{1}{b}\left(\frac{1}{x_{BP}} - \frac{1}{x_0}\right) & \text{for} \quad \rho = 1 , \\ \\ \frac{1}{ay_0(1-\rho)} \ln \; \rho + \frac{x_0}{x_{BP}}(1-\rho) & \text{for} \quad \rho \neq 1 , \end{cases} \tag{3.6.15}$$

where $\rho = bx_0/ay_0$ and t_{BP}^{X} is not defined for $0 \leq x_{BP} < x_0 - y_0 a/b$ when $\rho > 1$. When t_{BP}^{X} is not defined, we will set it equal to $(+\infty)$. Similarly,

$$t_{BP}^{Y} = \begin{cases} \frac{1}{a}\left(\frac{1}{y_{BP}} - \frac{1}{y_0}\right) & \text{for} \quad \rho = 1 , \\ \\ \frac{1}{ay_0(\rho-1)} \ln \; \frac{1}{\rho} + \frac{y_0}{y_{BP}}\left(1 - \frac{1}{\rho}\right) & \text{for} \quad \rho \neq 1 , \end{cases} \tag{3.6.16}$$

where t_{BP}^{Y} is not defined for $0 \leq y_{BP} < y_0 - x_0 b/a$ when $\rho < 1$. It follows [again with three cases to be considered] that

$$t_{BP}^{X} < t_{BP}^{Y} \text{ if and only if } a(y_0 - y_{BP}) > b(x_0 - x_{BP}) . \tag{3.16.17}$$

We observe that t_{BP}^{X} is well defined and finite when (3.6.17) holds and $x_{BP} > 0$. From the above we may conclude

PROPOSITION 3.6.2: When the combat dynamics are given by the LANCHESTER-type equations for an FT|FT attrition process (2.4.1), Y will win a fixed-force-level-breakpoint battle if and only if

$$\frac{x_0}{y_0} < \frac{a}{b} \left\{ \frac{1 - f_{BP}^{Y}}{1 - f_{BP}^{X}} \right\} . \tag{3.6.18}$$

The duration of combat is finite if and only if $f_{BP}^{X} > 0$, i.e. $x_{BP} > 0$.

Alternatively, we could have developed Proposition 3.6.2 by Method B. Again, we know that Condition (FI) is satisfied so that we can use (3.5.4), which is a special case of the victory-prediction condition in Proposition 3.5.1, to prove Proposition 3.6.2. To invoke (3.5.4), we observe that $H(x_0,x)$ (as given by (3.3.6)) is quasi-homogeneous and (3.3.10) yields that

$$\gamma(x_0) = x_0, \qquad \text{and} \qquad h_1(\frac{x}{x_0}) = b \left\{ 1 - (\frac{x}{x_0}) \right\}, \tag{3.6.19}$$

since $H(x_0,x) = b(x_0-x)$. Similarly,

$$\lambda(y_0) = y_0 , \qquad \text{and} \qquad k_1(\frac{y}{y_0}) = a \left\{ 1 - (\frac{y}{y_0}) \right\} . \tag{3.6.20}$$

Consequently, $\gamma(x_0)/\lambda(y_0) = \phi(x_0/y_0) = x_0/y_0$ so that $c = 1 > 0$, and we may invoke (3.5.4) to prove Proposition 3.6.2 provided that t_{BP}^{X} is finite. Thus, we must consider t_{BP}^{X} as given by (3.6.15). Proposition 3.6.2, however, requires that

$$\frac{a}{b} y_0 \geq \frac{a}{b} (y_0 - y_{BP}) > x_0 - x_{BP} , \tag{3.6.21}$$

so that (3.6.15) yields that t_{BP}^X is well defined and finite if and only if $x_{BP} > 0$. We finally note that the force-annihilation-prediction condition given in Proposition 2.4.1 may be obtained from Proposition 3.6.2 by setting $f_{BP}^X = f_{BP}^Y = 0$.

3.7. Development of General Battle-Outcome-Prediction Conditions for Deterministic LANCHESTER-Type Attrition Processes with Stochastic Casualty-Fraction Breakpoints.

In this case we consider that each side's force-level breakpoint is a random variable that is realized before the beginning of battle (see Section 3.4 above). Although this model may seem somewhat restrictive, it is equivalent (see Section 3.4 above and HELMBOLD [10, p. 5 and pp. 68-69]) to one in which each side considers its own force level and governs its behavior according to its own state, i.e. force level. We will denote random variables by upper-case letters, with their realizations being denoted by the corresponding lower-case letters. Thus, X_{BP} is a random variable (frequently abbreviated r.v.) and denotes X's force-level breakpoint. The realization of X_{BP} in a particular battle will be denoted as x_{BP} in consonance with out previous notation.

The outcome of battle is now a random variable that depends on both the deterministic battle dynamics and the distribution functions for the two force-level breakpoints. Quantities that are of interest for our combat model include the following:

(Q1) the probability of winning,

(Q2) the casualty-fraction distributions (both conditional and also unconditional),

and (Q3) the average casualty fraction for each side.

Let us first consider the probability of winning. We assume that the random variables X_{BP} and Y_{BP} are independent and continuous. Let us further assume that T^X_{BP} is finite, where T^X_{BP} denotes the time at which X_{BP} is reached. Invoking Proposition 3.3.1, we see that Y will win if and only if

$$X_{BP} > g(Y_{BP}) = g(Y_{BP};x_0,y_0) \ . \tag{3.7.1}$$

Hence, the probability that Y will win is given by $\mathrm{Prob}[X_{BP} > g(Y_{BP})]$, which we may write as $\mathrm{Prob}[X_{BP} \geq g(Y_{BP})]$, since by assumption $\mathrm{Prob}[X_{BP} = g(Y_{BP})] = 0$. For convenience, we will denote $\mathrm{Prob}[X_{BP} \geq g(Y_{BP})]$ as $P[X_{BP} \geq g(Y_{BP})]$. Although we could proceed to develop the desired results in terms of force levels or force-level fractions (e.g. x/x_0), it is more convenient to develop them in terms of the casualty-fractions, since the results that appear in the literature [10; 25] are expressed this way. Both approaches are, of course, equivalent as Proposition 3.3.3 and comparison of Propositions 3.3.1 and 3.3.1' shows us.

We now develop expressions for the probability that Y will win a battle with deterministic LANCHESTER-type combat dynamics and stochastic (or random) casualty-fraction breakpoints by considering Proposition 3.3.1'. Recalling that the X and Y casualty fractions are given by

$$f_c^X = \frac{x_0-x}{x_0} , \qquad \text{and} \qquad f_c^Y = \frac{y_0-y}{y_0} , \tag{3.7.2}$$

we will denote X's casualty-fraction breakpoint (a r.v.) as $(F_c^X)_{BP}$, with corresponding distribution function (d.f.) denoted as $F_X[(f_c^X)_{BP}]$. In other words, $F_X(s)$ denotes $P[(F_c^X)_{BP} \leq s]$. We will also denote the corresponding d.f. as $\bar{F}_X(s)$. In other words, $\bar{F}_X(s) = 1 - F_X(s)$. Furthermore, the probability that $(F_c^X)_{BP}$ lies between s and $s + ds$ is given by

$$P[s \leq (F_c^X)_{BP} \leq s + ds] = dF_X(s) \ . \tag{3.7.3}$$

We will also write $dF_X(s) = f_X(s)ds$, where $f_X(s)$ is called the probability density function (p.d.f.) of the random variable $(F_c^X)_{BP}$. From the

assumed independence of X_{BP} and Y_{BP}, it follows that $(F_c^X)_{BP}$ and $(F_c^Y)_{BP}$ are independent.

With the above notation defined, we may invoke Proposition 3.3.1' to find that the probability that Y will win is given by

$$P_Y = P[Y \text{ will win}] = P[(F_c^X)_{BP} < \varphi((F_c^Y)_{BP})] \ . \tag{3.7.4}$$

It follows from the assumed independence of $(F_c^X)_{BP}$ and $(F_c^Y)_{BP}$ (see Appendix B concerning the probability that one random variable is less than another independent one) that

$$P_Y = \int_0^1 F_X(\psi(t)) \ dF_Y(t) \ , \tag{3.7.5}$$

where we have truncated $\varphi(t)$ by defining (see HELMBOLD [10, pp. 12-13] for a further discussion)

$$\psi(t) = \text{Minimum}[\varphi(t),1] \ . \tag{3.7.6}$$

Moreover, the probability that Y will win may also be written as

$$P_Y = P[\psi^{-1}((F_c^X)_{BP}) < (F_c^Y)_{BP}] \ , \tag{3.7.7}$$

so that we also have

$$P_Y = \int_0^1 \overline{F}_Y(\psi^{-1}(s)) \ dF_X(s) \ . \tag{3.7.8}$$

In the above formulas (3.7.5) and (3.7.8), the variables s and t are thus related by

$$s = \psi(t) \ . \tag{3.7.9}$$

In order to preserve the correctness of the above formulas when $\varphi(1) < 1$, we define $\psi^{-1}(s) = 1$ for $\varphi(1) \leq s \leq 1$ (<u>see</u> HELMBOLD [10, p. 13]). Equation (3.7.8) also follows from (3.7.5) by an integration by parts and the change of variable (3.7.9). In a similar fashion, it may be shown that

$$P_X = \int_0^1 F_Y(\psi^{-1}(s))\ dF_X(s) = \int_0^1 \overline{F}_X(\psi(t))\ dF_Y(t) \ . \tag{3.7.10}$$

The development of expressions for the casualty-fraction conditional distributions[15] is somewhat more involved. We begin by considering the event that a battle is fought and we observe $s \leq (F_c^X)_{BP} \leq s + ds$. The probability that this happens in any battle is given by

$$P[s \leq (F_c^X)_{BP} \leq s + ds] = \overline{F}_Y(\psi^{-1}(s))\ dF_X(s) \ . \tag{3.7.11}$$

It follows that

$$P[(F^X)_{BP} \leq p] = \int_0^p \overline{F}_Y(\psi^{-1}(s))\ dF_X(s) \ . \tag{3.7.12}$$

Consequently, the probability that this happens in a battle won by Y is given by

$$P[(F_c^X)_{BP} \leq p | Y \ \text{wins}] = \frac{1}{P_Y} \int_0^p \overline{F}_Y(\psi^{-1}(s))\ dF_X(s) \ . \tag{3.7.13}$$

In a battle won by Y, however, we have $(F_c^X)_{BP} = (F_c^X)_f$, where $(F_c^X)_f$ denotes Y's final casualty fraction at the end of battle. Hence,

$$P[(F_c^X)_f \leq p \mid Y \text{ wins}] = \frac{1}{P_Y}\int_0^p \bar{F}_Y(\psi^{-1}(s))\, dF_X(s) \; . \tag{3.7.14}$$

In such a battle, the probability that $(F_c^Y)_f \leq q$ is the same as that $(F_c^Y)_f = \psi^{-1}((F_c^X)_f) \leq q$ or $(F_c^X)_f \leq \psi(q)$, since $f_c^X = \psi(f_c^Y)$. It follows that

$$P[(F_c^Y)_f \leq q \mid Y \text{ wins}] = \frac{1}{P_Y}\int_0^{\psi(q)} \bar{F}_Y(\psi^{-1}(s))\, dF_X(s) \; . \tag{3.7.15}$$

Thus, we have developed essentially all the results shown in Table 3.I except for the unconditional casualty-fraction distributions and the average casualty fractions.

To develop the expressions for the average casualty fractions given in Table 3.I, we first develop the distribution of, for example, X's final casualty fraction, denoted as $(F_c^X)_f$, and then simply compute its expected value. Since either X or Y must win, the law of total probability yields

$$P[(F_c^X)_f \leq p] = P[(F_c^X)_f \leq p \mid X \text{ wins}]\cdot P_X + P[(F_c^X)_f \leq p \mid Y \text{ wins}]\cdot P_Y \; , \tag{3.7.16}$$

so that (3.7.14) and the X analogue of (3.7.15) yield

$$P[(F_c^X)_f \leq p] = \int_0^{\psi^{-1}(p)} \bar{F}_X(\psi(t))\, dF_Y(t) + \int_0^p \bar{F}_Y(\psi^{-1}(s))\, dF_X(s) \; . \tag{3.7.17}$$

Recalling that $t = \psi^{-1}(s)$, we may change the variable of integration in the first term on the right-hand side of (3.7.17) to obtain

$$\int_0^{\psi^{-1}(p)} \bar{F}_X(\psi(t))\, dF_Y(t) = \int_0^p \bar{F}_X(s)\, dF_Y(\psi^{-1}(s)) \; ,$$

TABLE 3.I. Quantities of Interest for Battle with Random Casualty-Fraction Breakpoints.

1. Probabilities of Winning

$$P_X = \int_0^1 F_Y(\psi^{-1}(s))\, dF_X(s) = \int_0^1 \bar{F}_X(\psi(t))\, dF_Y(t)$$

$$P_Y = \int_0^1 F_X(\psi(t))\, dF_Y(t) = \int_0^1 \bar{F}_Y(\psi^{-1}(s))\, dF_X(s)$$

2a. Casualty-Fraction Conditional Distributions

$$P[(F_c^X)_f \leq p \mid X \text{ wins}] = \frac{1}{P_X} \int_0^{\psi^{-1}(p)} \bar{F}_X(\psi(t))\, dF_Y(t)$$

$$P[(F_c^Y)_f \leq q \mid X \text{ wins}] = \frac{1}{P_X} \int_0^q \bar{F}_X(\psi(t))\, dF_Y(t) = P[(F_c^Y)_{BP} \leq q \mid X \text{ wins}]$$

$$P[(F_c^X)_f \leq p \mid Y \text{ wins}] = \frac{1}{P_Y} \int_0^p \bar{F}_Y(\psi^{-1}(s))\, dF_X(s) = P[(F_c^X)_{BP} \leq p \mid Y \text{ wins}]$$

$$P[(F_c^Y)_f \leq q \mid Y \text{ wins}] = \frac{1}{P_Y} \int_0^{\psi(q)} \bar{F}_Y(\psi^{-1}(s))\, dF_X(s)$$

2b. Casualty-Fraction Distributions

$$P[(F_c^X)_f \leq p] = 1 - \bar{F}_X(p)\, \bar{F}_Y(\psi^{-1}(p))$$

$$P[(F_c^Y)_f \leq q] = 1 - \bar{F}_X(\psi(q))\, \bar{F}_Y(q)$$

3. Average Casualty Fractions

$$\bar{f}_c^X = \int_0^1 \bar{F}_X(s)\, \bar{F}_Y(\psi^{-1}(s))\, ds$$

$$\bar{f}_c^Y = \int_0^1 \bar{F}_X(\psi(t))\, \bar{F}_Y(t)\, dt = \int_0^1 \bar{F}_X(s)\, \bar{F}_Y(\psi^{-1}(s))\, \frac{ds}{\psi'(\psi^{-1}(s))}$$

whence (3.7.17) becomes

$$P[(F_c^X)_f \leq p] = -\int_0^p d\{\bar{F}_X(s)\ \bar{F}_Y(\psi^{-1}(s))\}\ . \tag{3.7.18}$$

Integration of the above then yields the desired result for the casualty-fraction distribution

$$P[(F_c^X)_f \leq p] = 1 - \bar{F}_X(p)\ \bar{F}_Y(\psi^{-1}(p))\ . \tag{3.7.19}$$

Let us also observe that[16]

$$P[(F_c^X)_f \geq p] = \bar{F}_X(p)\ \bar{F}_Y(\psi^{-1}(p))\ . \tag{3.7.20}$$

From (3.7.18) we see that the expected value of $(F_c^X)_f$ is given by

$$\bar{f}_c^X = -\int_0^1 s\ d\{\bar{F}_X(s)\ \bar{F}_Y(\psi^{-1}(s))\}\ ,$$

whence an integration by parts yields the desired result given in Table 3.I. The expressions for $P[(F_c^Y)_f \leq q]$ and $\bar{f}_c^Y$ may be developed in a similar fashion.

Thus, we have developed general expressions for the three measures of combat outcomes: (1) the probability of winning, (2) the casualty-fraction distributions, and (3) the average casualty fraction for each side. We did this for the case in which (1) combat attrition was modelled by deterministic LANCHESTER-type equations for which Condition (CFI) held, and (2) the casualty-fraction breakpoint for each side was a random variable independent of the other side's breakpoint and with known distribution. Unfortunately, the general combat-outcome-prediction

expressions given in Table 3.I do not by themselves provide any insight (such as that provided by Proposition 3.6.1 discussed in Sections 3.6 and 2.8 above) into the relationship between the distribution of combat outcomes and various factors in the combat model (such as the initial force ratio x_0/y_0, relative fire effectiveness, parameters of the breakpoint distributions, etc.). To develop such parametric insights into the distribution of combat outcomes, we must consider some specific instances.

One general case, however, for which fairly explicit results arise is that in which[17]

$$\bar{F}_X(s) = [\bar{F}_Y(t)]^a , \tag{3.7.21}$$

where $s = \psi(t)$, $\psi(t)$ is given by (3.7.6), and, of course, Condition (CFI) holds. Such a case has been found to be a reasonably good approximation to U. S. Civil War data by H. K. WEISS [25]. To develop an explicit expression for the probability that Y wins, we let $\tau = \bar{F}_Y(t)$ so that $dF_Y(t) = -\,d\tau$ and $\bar{F}_X(\psi(t)) = \tau^{1/a}$, and then (3.7.5) yields

$$P_Y = \int_1^0 \tau^{1/a}(-d\tau) = \int_0^1 \tau^{1/a}\, d\tau,$$

or

$$P_Y = \frac{a}{a+1} . \tag{3.7.22}$$

To develop an explicit expression for the conditional distribution of, for example, X's (final) casualty fraction (3.7.14), we let

$\sigma = \overline{F}_X(s)$ so that $dF_X(s) = -\,d\sigma$ and $\overline{F}_Y(\psi^{-1}(s)) = \sigma^{1/a}$, and then (3.7.14) yields

$$P[(F_c^X)_f \le p \,|\, Y \text{ wins}] = \left(\frac{a+1}{a}\right) \int_1^{\overline{F}_X(p)} \sigma^{1/a}(-d\sigma) \; ,$$

or

$$P[(F_c^X)_f \le p \,|\, Y \text{ wins}] = 1 - [\overline{F}_X(p)]^{(1+1/a)} \; . \tag{3.7.23}$$

Other results may be obtained in a similar fashion, and these results are summarized in Table 3.II. From these results, we see that the assumption (3.7.21) has the implication that whether a side wins or loses does not affect his casualty-fraction distribution, i.e. the unconditional casualty-fraction distribution is the same as the conditional distributions.

We are now in a position to develop some explicit results for LANCHESTER's two classic combat formulations: (1) LANCHESTER's equations for area fire (2.4.1), and (2) LANCHESTER's equations for modern warfare (2.2.1). Here we will give exact results when they are relatively simple (i.e. for the case in which each side's casualty-fraction breakpoint is a uniformly distributed random variable) and will use an approximation when the results are not simple (i.e. for the special case in which each side's casualty-fraction breakpoint is an exponentially distributed random variable).

TABLE 3.II. Results for Battle with Random Casualty-Fraction Breakpoints When $\bar{F}_X(s) = [\bar{F}_Y(t)]^a$.

Basic Assumption: $\bar{F}_X(s) = [\bar{F}_Y(t)]^a$ where $s = \psi(t)$.

Probabilities of Winning:

$$P_X = \frac{1}{a+1}, \qquad P_Y = \frac{a}{a+1}$$

Casualty-Fraction Conditional Distributions:

$$P[(F_c^X)_f \le p \mid X \text{ wins}] = 1 - [\bar{F}_Y(\psi^{-1}(p))]^{(a+1)}$$

$$P[(F_c^Y)_f \le q \mid X \text{ wins}] = 1 - [\bar{F}_Y(q)]^{(a+1)}$$

$$P[(F_c^X)_f \le p \mid Y \text{ wins}] = 1 - [\bar{F}_X(p)]^{(1+1/a)}$$

$$P[(F_c^Y)_f \le q \mid Y \text{ wins}] = 1 - [\bar{F}_X(\psi(q))]^{(1+1/a)}$$

Casualty-Fraction Distributions:

$$P[(F_c^X)_f \le p] = 1 - [\bar{F}_X(p)]^{(1+1/a)} = P[(F_c^X)_f \le p \mid X \text{ wins}]$$

$$P[(F_c^Y)_f \le q] = 1 - [\bar{F}_Y(q)]^{(a+1)} = P[(F_c^Y)_f \le q \mid Y \text{ wins}]$$

Average Casualty Fractions:

$$\bar{f}_c^X = \int_0^1 [\bar{F}_X(s)]^{(1+1/a)}\, ds, \qquad \bar{f}_c^Y = \int_0^1 [\bar{F}_Y(t)]^{(1+a)}\, dt$$

3.8. Battle-Outcome-Prediction Conditions for Deterministic FT|FT Attrition Process with Stochastic Breakpoints

In this section we develop explicit expressions for (Q1) the probability of winning, (Q2) the casualty-fraction distributions (both conditional and also unconditional), and (Q3) the average casualty fraction for each side for LANCHESTER's (deterministic) equations for area fire (2.4.1), i.e. the equations for an FT|FT attrition process, with random breakpoints. We will do this for two specific casualty-fraction-breakpoint distributions:

(D1) uniformly distributed breakpoints,

and (D2) exponentially distributed breakpoints.

As above, we will assume that each side's breakpoint is independent of that for the other side. Let us observe that for random breakpoints the analogue of the victory-prediction condition (3.6.18) is a probability of winning such as (3.8.6) below. Also, the analogue of the victor's casualty fraction given in Table 2.XII is, for example, a casualty-fraction conditional distribution such as (3.8.7) or an average loss such as given by (3.8.10).

We begin by developing certain key general expressions that apply to all casualty-fraction-breakpoint distributions for an FT|FT attrition process. First, let us observe that we may express the state equation (2.4.3) for the FT|FT attrition equations (2.4.1) in terms of the casualty fractions f_c^X and f_c^Y [see (3.7.2)] as

$$f_c^X = \gamma f_c^Y , \tag{3.8.1}$$

where

$$\gamma = \frac{a}{b} \cdot \left(\frac{y_0}{x_0}\right) \tag{3.8.2}$$

Moreover, there are restrictions on the applicability of (3.8.1), i.e., it holds only for f_c^X, $f_c^Y \in [0,1]$. We observe that $\gamma > 0$ is simply the ratio of fractional losses (X to Y). In other words, from (3.8.1) we see that the function φ of Condition (CFI), i.e. the function such that $f_c^X = \varphi(f_c^Y)$, is given by

$$\varphi(t) = \gamma t \ . \tag{3.8.3}$$

Since $f_c^X \le 1$, we must sometimes truncate the φ function (i.e. when $t > 1/\gamma$) so that f_c^X does not exceed 1. Thus, we introduce the modified function ψ defined by (3.7.6). It is given by

$$\psi(t) = \begin{cases} \gamma t & \text{for} \quad 0 \le t \le 1/\gamma \ , \\ 1 & \text{for} \quad 1/\gamma \le t \ . \end{cases} \tag{3.8.4}$$

We also observe that

$$\psi^{-1}(s) = \begin{cases} s/\gamma & \text{for} \quad 0 \le s \le \gamma \ , \\ 1 & \text{for} \quad \gamma \le s \ . \end{cases} \tag{3.8.5}$$

Next, we will use the above in conjunction with the general expressions given in Table 3.I to develop the key general battle-outcome-prediction expressions for an FT|FT attrition process. Considering (3.8.5), we obtain from (3.7.8) the following expression for P_Y

$$P_Y = \begin{cases} \int_0^{\gamma} \overline{F}_Y(\frac{s}{\gamma})\ dF_X(s) & \text{for} \quad 0 \le \gamma \le 1 \ , \\ \int_0^{1} \overline{F}_Y(\frac{s}{\gamma})\ dF_X(s) & \text{for} \quad 1 \le \gamma \ . \end{cases} \tag{3.8.6}$$

In a similar fashion, consideration of (3.8.5), (3.8.6), and (3.7.13) yields

$$P[(F_c^X)_f \le p \mid Y \text{ wins}] = \begin{cases} 1 & \text{for } 0 \le \gamma \le p, \\ \{\int_0^p \bar{F}_Y(\frac{s}{\gamma})dF_X(s)\}/\{\int_0^\gamma \bar{F}_Y(\frac{s}{\gamma})dF_X(s)\} & \text{for } p \le \gamma \le 1, \\ \{\int_0^p \bar{F}_Y(\frac{s}{\gamma})dF_X(s)\}/\{\int_0^1 \bar{F}_Y(\frac{s}{\gamma})dF_X(s)\} & \text{for } 1 \le \gamma . \end{cases} \tag{3.8.7}$$

The other conditional casualty-fraction distributions may be similarly obtained. For the unconditional casualty fraction distributions, it is more convenient to consider the complementary d.f. Hence, (3.7.20) and (3.8.5) yields

$$P[(F_c^X)_f \ge p] = \begin{cases} 0 & \text{for } 0 \le \gamma \le p, \\ \bar{F}_X(p)\ \bar{F}_Y(\frac{p}{\gamma}) & \text{for } p \le \gamma . \end{cases} \tag{3.8.8}$$

Similarly,

$$P[(F_c^Y)_f \ge q] = \begin{cases} \bar{F}_X(\gamma q)\ \bar{F}_Y(q) & \text{for } 0 \le \gamma \le 1/q , \\ 0 & \text{for } 1/q \le \gamma . \end{cases} \tag{3.8.9}$$

Finally, Table 3.I, (3.8.4), and (3.8.5) yield the desired expression for the average losses,

$$\bar{f}_c^X = \gamma \bar{f}_c^Y = \begin{cases} \int_0^{\gamma} \bar{F}_X(s)\, \bar{F}_Y(\frac{s}{\gamma})\, ds & \text{for} \quad 0 \le \gamma \le 1, \\ \int_0^{1} \bar{F}_X(s)\, \bar{F}_Y(\frac{s}{\gamma})\, ds & \text{for} \quad 1 \le \gamma. \end{cases} \tag{3.8.10}$$

From the above general battle-outcome-prediction conditions it is straightforward (but sometimes very messy) to compute desired quantities for a specific casualty-fraction-breakpoint distribution.

We now will consider two such specific breakpoint distributions. Let us first consider the case in which each side's casualty-fraction breakpoint is uniformly distributed (i.e. a side is equally likely to break off the engagement at any casualty fraction between 0 and 1) and, of course, the battle dynamics are given by the equations for an FT|FT attrition process (2.4.1). In this case

$$F_X(s) = s \ , \qquad \text{and} \qquad F_Y(t) = t \ , \tag{3.8.11}$$

for $0 \le s, t \le 1$. Use of (3.8.11) in formulas like (3.8.6) through (3.8.10) then yields the results given in Table 3.III.

Before we proceed with the development of results for exponentially distributed breakpoints, let us consider what insights into the dynamics of combat we can obtain from our simple combat model. In particular (as stressed above), we are interested in understanding the structure of the relationship between battle outcome and values for model parameters. For this case in which we have introduced randomness into the two (independent) processes of breaking off the engagement, we would like to know the effects on the nature of battle outcomes from introducing this randomness. The reader

TABLE 3.III. Results for Battle with FT|FT Attrition and Uniformly Distributed Casualty-Fraction Breakpoints

Probabilities of Winning:

$$P_X = \begin{cases} 1 - \gamma/2 & \text{for } 0 \le \gamma \le 1 \\ 1/(2\gamma) & \text{for } 1 \le \gamma, \end{cases} \qquad P_Y = \begin{cases} \gamma/2 & \text{for } 0 \le \gamma \le 1 \\ 1 - 1/(2\gamma) & \text{for } 1 \le \gamma, \end{cases}$$

where $\gamma = (a/b)(y_0/x_0)$.

Casualty-Fraction Conditional Distributions:

$$P[(F_c^X)_f \le p \mid X \text{ wins}] = \begin{cases} 1 & \text{for } 0 \le \gamma \le p \\ \dfrac{1 - (1-p)^2}{\gamma(2-\gamma)} & \text{for } p \le \gamma \le 1 \\ 1 - (1-p)^2 & \text{for } 1 \le \gamma, \end{cases} \qquad P[(F_c^Y)_f \le q \mid X \text{ wins}] = \begin{cases} \dfrac{1 - (1-q\gamma)^2}{\gamma(2-\gamma)} & \text{for } 0 \le \gamma \le 1 \\ 1 - (1-q\gamma)^2 & \text{for } 1 \le \gamma \le 1/q \\ 1 & \text{for } 1/q \le \gamma \end{cases}$$

$$P[(F_c^X)_f \le p \mid Y \text{ wins}] = \begin{cases} 1 & \text{for } 0 \le \gamma \le p \\ 1 - (1 - \frac{p}{\gamma})^2 & \text{for } p \le \gamma \le 1 \\ \dfrac{\gamma^2(1 - (1 - \frac{p}{\gamma})^2)}{(2\gamma - 1)} & \text{for } 1 \le \gamma, \end{cases} \qquad P[(F_c^Y)_f \le q \mid Y \text{ wins}] = \begin{cases} 1 - (1-q)^2 & \text{for } 0 \le \gamma \le 1 \\ \dfrac{\gamma^2(1 - (1-q)^2)}{(2\gamma-1)} & \text{for } 1 \le \gamma \le 1/q \\ 1 & \text{for } 1/q \le \gamma \end{cases}$$

Casualty-Fraction Distributions:

$$P[(F_c^X)_f \ge p] = \begin{cases} 0 & \text{for } 0 \le \gamma \le p \\ (1-p)(1 - \frac{p}{\gamma}) & \text{for } p \le \gamma, \end{cases} \qquad P[(F_c^Y)_f \ge q] = \begin{cases} (1-\gamma q)(1-q) & \text{for } 0 \le \gamma \le 1/q \\ 0 & \text{for } 1/q \le \gamma \end{cases}$$

Average Casualty Fractions:

$$\bar{f}_c^X = \gamma \bar{f}_c^Y = \begin{cases} \frac{\gamma}{2}(1 - \frac{\gamma}{3}) & \text{for } 0 \le \gamma \le 1 \\ \frac{1}{2}(1 - \frac{1}{3\gamma}) & \text{for } 1 \le \gamma \end{cases}$$

should therefore compare the deterministic-breakpoint results given in Table 2.XII with the uniformly-distributed-breakpoint results given in Table 3.III. Let us now make a few such comparisons.

In Figure 3.5 we show the probability that Y will win, P_Y, as a function of the "normalized" initial force ratio, $\gamma = (a/b)(y_0/x_o)$. Let us recall (<u>see</u> (2.8.14) or Proposition 3.6.2) that for deterministic breakpoints <u>Y will win a fixed-casualty-fraction-breakpoint battle if and only if</u>

$$\frac{x_0}{y_0} < \frac{a}{b}\frac{(f_c^Y)_{BP}}{(f_c^X)_{BP}}, \qquad (3.8.12)$$

with the length of battle being finite if and only if $(f_c^X)_{BP} < 1$. Thus, for equal breakpoints, i.e. $(f_c^X)_{BP} = (f_c^Y)_{BP}$, Y will win if and only if $\gamma > 1$. In other words, for deterministic breakpoints Y will win with probability one for $\gamma > 1$ and will win with probability zero for $0 \leq \gamma < 1$ (<u>see</u> Figure 3.6). Hence, we see that the normalized initial force ratio, $\gamma = (a/b)(y_0/x_0)$, is the key parameter for forecasting whether Y will win or lose for both deterministic breakpoints and also random ones.

We may think of $(F_c^X)_f$ as the force-level cost to X of engaging Y in combat without considering the outcome of battle, i.e. without considering whether X wins or loses. Then the casualty-fraction distributions may be considered measures of the risk of doing battle. In Table 3.III let us note that for $\gamma \neq 1$

$$P[(F_c^X)_f \leq p | X \text{ wins}] \neq P[(F_c^Y)_f \leq p | Y \text{ wins}], \qquad (3.8.13)$$

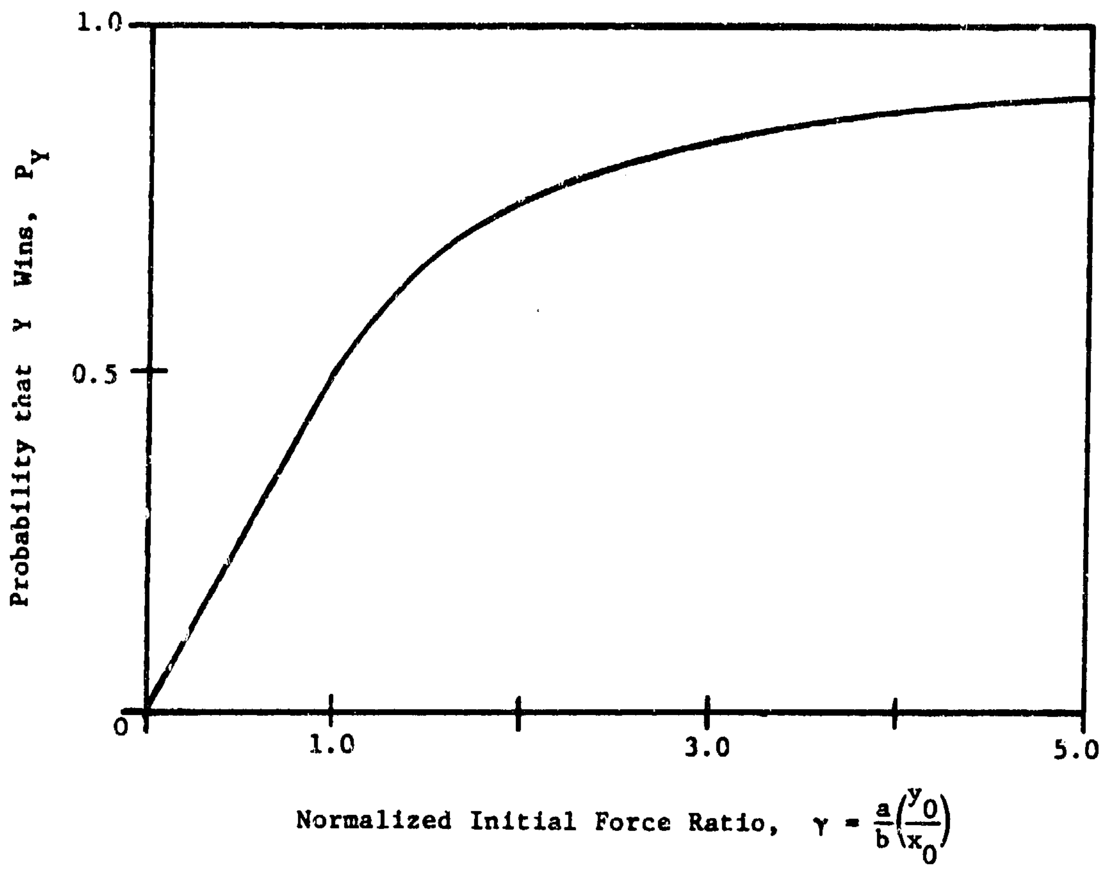


Figure 3.5. Relationship between the normalized initial force ratio $\gamma = \frac{a}{b}\left(\frac{y_0}{x_0}\right)$ and the probability of winning for battle with deterministic FT|FT attrition and uniformly distributed breakpoints.

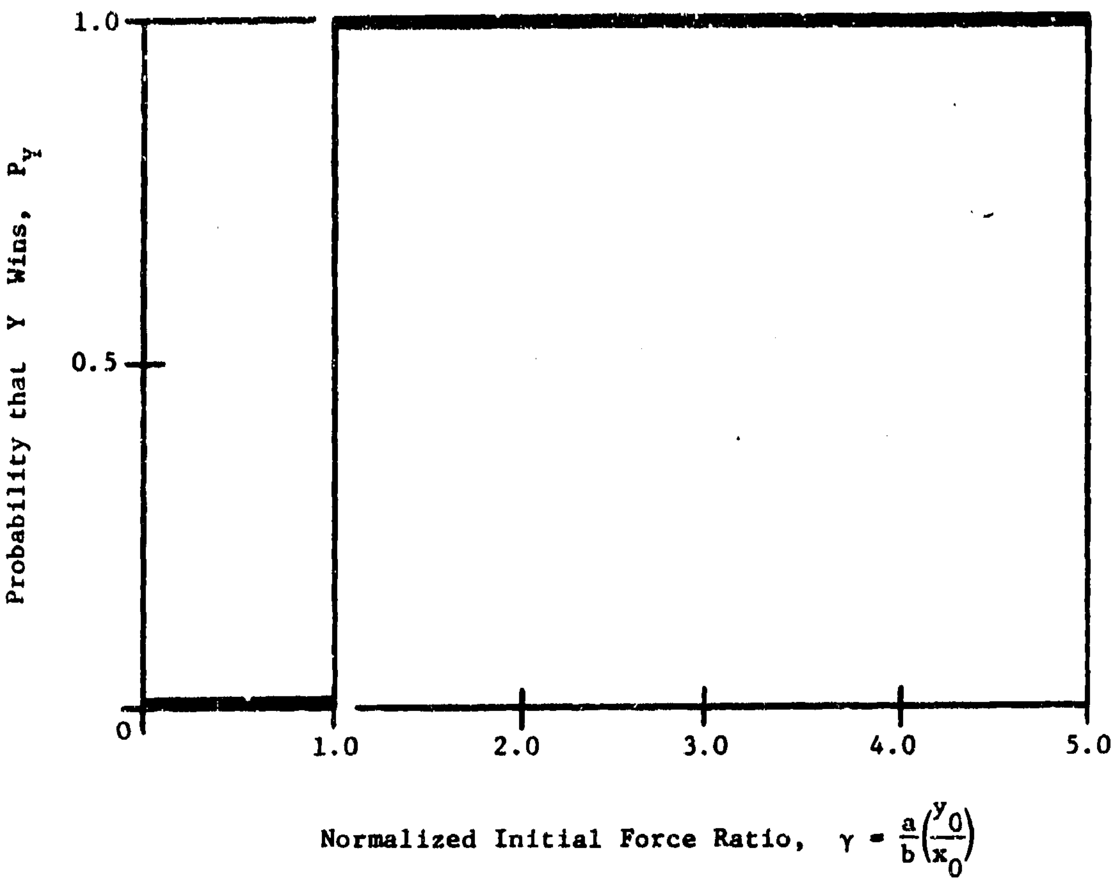


Figure 3.6. Relationship between the normalized initial force ratio $\gamma = \frac{a}{b}\left(\frac{y_0}{x_0}\right)$ and the probability of winning for battle with deterministic FT|FT attrition and deterministic equal breakpoints, e.g. $(f_c^X)_{BP} = (f_c^Y)_{BP}$.

and also that for $\gamma \neq 1$

$$P[(F_c^X)_f \leq p \mid Y \text{ wins}] \neq P[(F_c^Y)_f \leq p \mid X \text{ wins}] \ . \tag{3.8.14}$$

Hence, for $\gamma \neq 1$ the distribution of X's casualties when X wins is not the same as that of Y when Y wins, and (similarly) the distribution of X's casualties when Y wins is not the same as that of Y when X wins. It may be shown (see HELMBOLD [10, pp. 18-19]) that these results hold in general for FT|FT attrition. We will return to these results, i.e. (3.8.13) and (3.8.14), later, since they have an important role to play in the historical validation of such breakpoint hypotheses.

In Figure 3.7 we show how $P[(F_c)_f \leq 0.3]$, where $(F_c)_f$ denotes a given side's final casualty fraction, depends on the normalized initial force ratio γ and the outcome of battle. It should be clear that the curves shown in Figure 3.7 reflect the fact that (3.8.13) and (3.8.14) hold, e.g. the winner's casualty-fraction distribution is different for X and Y. Finally, in Figure 3.8 we show plots of the probability that X's final casualty fraction exceeds a given amount p as a function of the normalized initial force ratio γ. Also shown is the probability that X wins. Recalling the interpretation of $(F_c^X)_f$ as the cost to X of engaging Y (without considering the outcome of battle), we may think of Figure 3.8 as showing the risk to X of engaging Y in combat. We should observe that $P[(F_c^X)_f \geq p]$ quickly reaches a limiting value for γ greater than one. Let us finally observe that for any breakpoint distribution (see 3.8.10) the average casualty fractions are related by

$$\bar{f}_c^X = \gamma \bar{f}_c^Y \ , \tag{3.8.15}$$

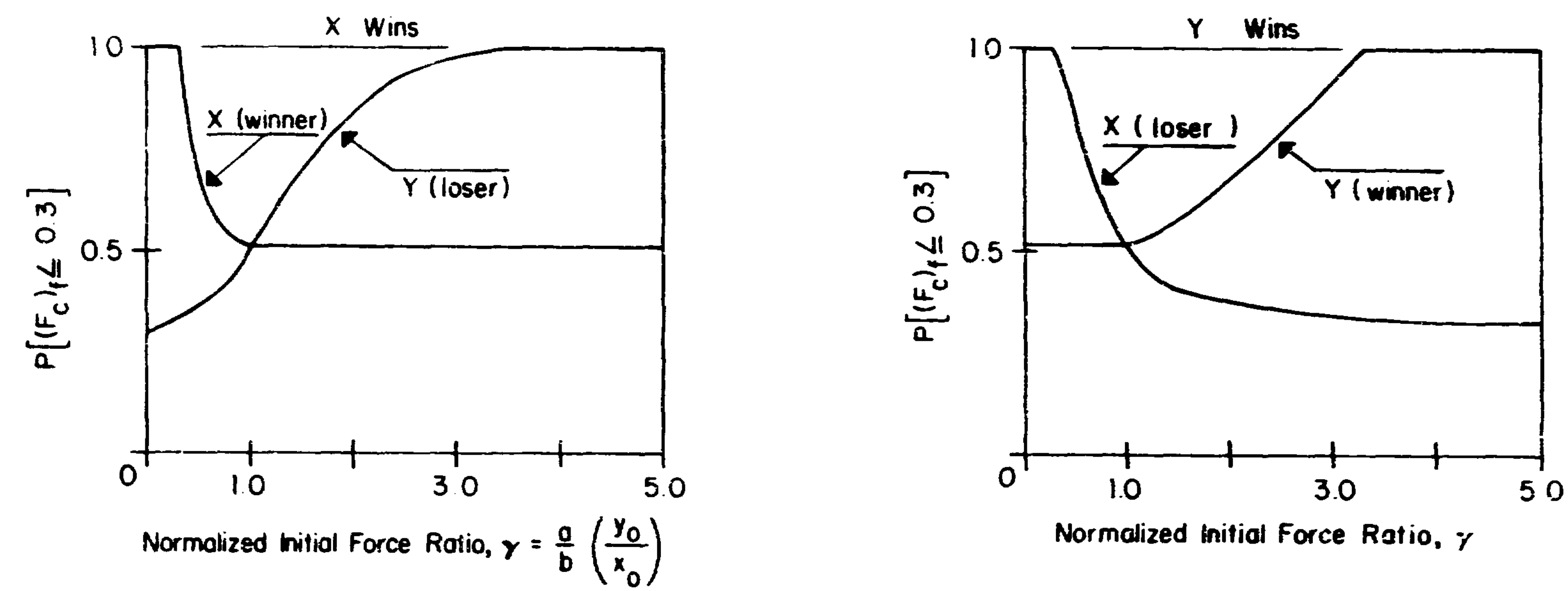


Figure 3.7. The probability that a side's final casualty fraction is less than or equal to 0.3 as a function of the normalized initial force ratio, $\gamma = \frac{a}{b}\left(\frac{x_0}{y_0}\right)$, and the engagement's outcome for a battle with deterministic FT|FT attrition and uniformly distributed breakpoints.

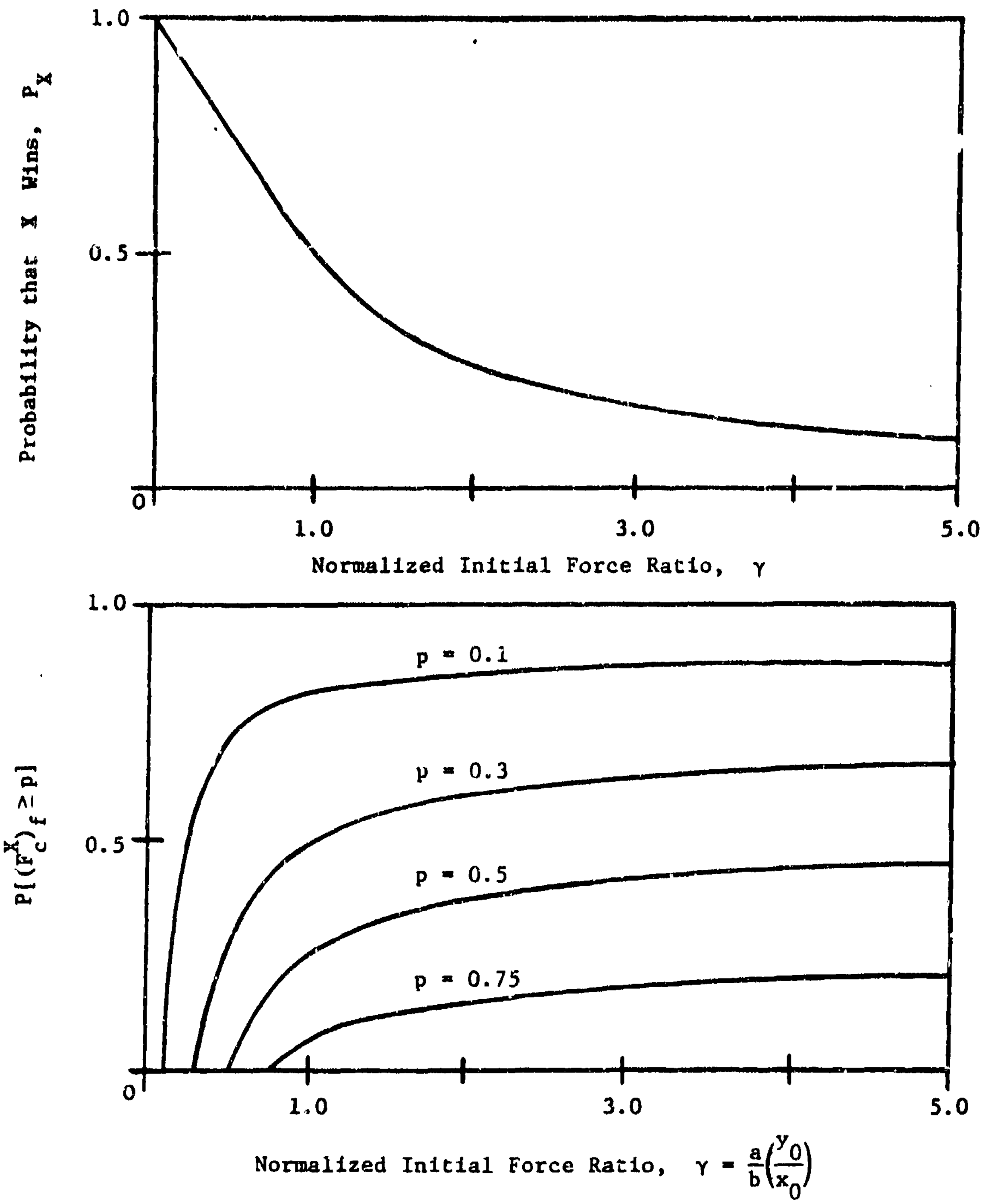


Figure 3.8. The probability that X's casualty fraction exceeds a given amount as a function of the normalized initial force ratio $\gamma = \frac{a}{b}\left(\frac{y_0}{x_0}\right)$ for battle with deterministic FT|FT attrition and uniformly distributed breakpoints.

which should be compared to the corresponding result for deterministic breakpoints (<u>see</u> Table 2.XII or (3.8.1))

$$f_c^X = \gamma f_c^T . \tag{3.8.16}$$

Thus, on the average the two sides' fractional losses bear the same relationship to each other whether or not the breakpoints are modelled as random variables.

Finally, we consider the case in which each side's <u>casualty-fraction breakpoint</u> is <u>exponentially distributed</u> and, of course, the battle dynamics are given by the equations for an FT|FT attrition process (2.4.1). In this case

$$F_X(s) = \frac{1 - e^{-\lambda_X s}}{1 - e^{-\lambda_X}} , \qquad \text{and} \qquad F_Y(t) = \frac{1 - e^{-\lambda_Y t}}{1 - e^{-\lambda_Y}} , \tag{3.8.17}$$

for $0 \leq s, t \leq 1$. We have added, for example, the factor $(1 - e^{-\lambda_X})$ to cure the defect of the distribution[18] $1 - e^{-\lambda_X s}$ at $s = 1$, i.e. to make $F_X(1) = 1$. The two distribution functions (3.8.11) and (3.8.17) for X's casualty-fraction breakpoint are graphically depicted in Figure 3.9. The parameter λ_X in (3.8.17) controls X's rate of "giving up his mission and breaking off the engagement" as a function of his casualty fraction. In other words, the larger λ_X is, the more "quickly" X gives up (as a function of his fractional loss) as computation of X's average breakpoint shows

$$(\bar{f}_c^X)_{BP} = E[(F_c^X)_{BP}] = \frac{1}{\lambda_X} - \frac{1}{(e^{\lambda_X} - 1)} , \tag{3.8.18}$$

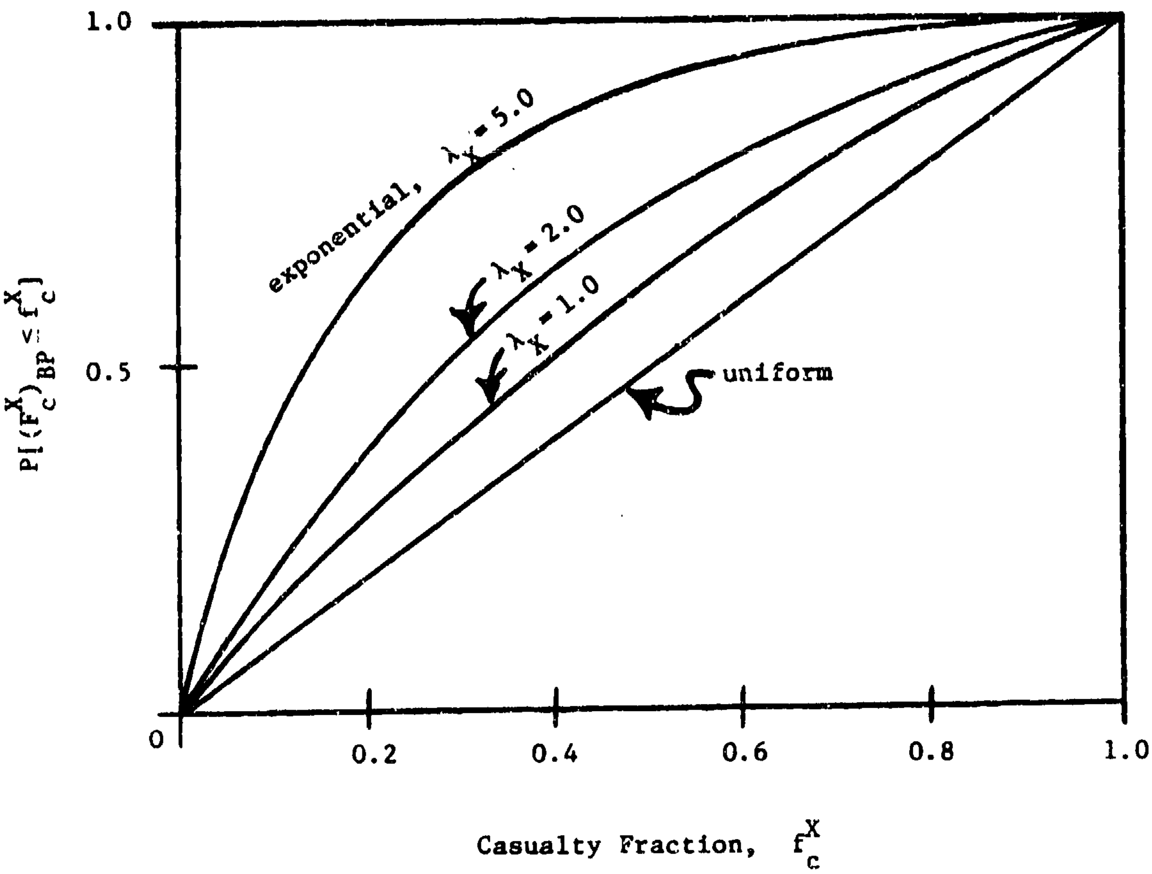


Figure 3.9. Uniform and exponential theoretical distribution functions for X's casualty-fraction breakpoint.

where $E[X]$ denotes the expected value of the random variable X. Table 3.IV shows how X's average breakpoint depends on the parameter λ_X.

TABLE 3.IV. X's Average Breakpoint $(\bar{f}_c^X)_{BP}$ as a Function of the Parameter λ_X for an Exponentially Distributed Casualty-Fraction Breakpoint.

	$\lambda_X = 1.0$	$\lambda_X = 2.0$	$\lambda_X = 5.$	$\lambda_X = 10.0$
$(\bar{f}_c^X)_{BP}$:	0.418	0.343	0.192	0.100

Exact results for the above exponential distribution functions (3.8.17) (see HELMBOLD [10, pp. 78-82]) are so complicated that it is difficult to clearly see the relationship between values for model parameters and battle outcome. For example, the probability that Y will win is readily computed from (3.8.6) and (3.8.17) to be given by

$$P_Y = \begin{cases} \dfrac{1}{(1-e^{-\lambda_X})(1-e^{-\lambda_Y})} \left\{ \dfrac{(\lambda_X/\lambda_Y)}{[(\lambda_X/\lambda_Y) + 1/\gamma]} - e^{-\lambda_Y} \left[1 - \dfrac{e^{-\lambda_X \gamma}}{(1 + \gamma\lambda_X/\lambda_Y)} \right] \right\} & \text{for } 0 \le \gamma \le 1, \\ \left\{ \dfrac{(\lambda_X/\lambda_Y)}{[(\lambda_X/\lambda_Y) + 1/\gamma]} \right\} \dfrac{\{1 - e^{-(\lambda_X + \lambda_Y/\gamma)}\}}{(1-e^{-\lambda_X})(1-e^{-\lambda_Y})} - \dfrac{e^{-\lambda_Y}}{(1-e^{-\lambda_Y})} & \text{for } 1 \le \gamma . \end{cases} \tag{3.8.19}$$

However, for $\lambda_X, \lambda_Y \geq 5.0$, P_Y is very nearly given by

$$\hat{P}_Y = \frac{(\lambda_X/\lambda_Y)}{[(\lambda_X/\lambda_Y) + 1/\gamma]} , \tag{3.8.20}$$

where $\hat{P}_Y$ denotes an approximate probability that Y wins.

Recalling (3.7.22), the result (3.8.20) suggests using the following approximations for the breakpoint distribution functions for $\lambda_X, \lambda_Y \geq 5.0$

$$\hat{F}_X(s) = 1 - e^{-\lambda_X s}, \qquad \text{and} \qquad \hat{F}_Y(t) = 1 - e^{-\lambda_Y t} , \tag{3.8.21}$$

and taking

$$s = \gamma t \qquad \text{for all} \quad t \geq 0 . \tag{3.8.22}$$

In this case

$$\hat{\bar{F}}_X(s) = e^{-\lambda_X s} = [e^{-\lambda_X s}]^{\gamma\lambda_X/\lambda_Y} = [\hat{\bar{F}}_Y(t)]^a , \tag{3.8.23}$$

where

$$a = \gamma\lambda_X/\lambda_Y , \tag{3.8.24}$$

so that (3.7.22) would yield (3.8.20). Figure 3.10 shows X's exact and approximate breakpoint distribution functions for $\lambda_X = 2.0$. For $\lambda_X = 5.0$, the approximate and exact values differ by at most 0.007 at $s = 1$: in other words, we would not be able to see any difference between them in a plot like Figure 3.10. Hence, our advocacy of the approximations (3.8.21)

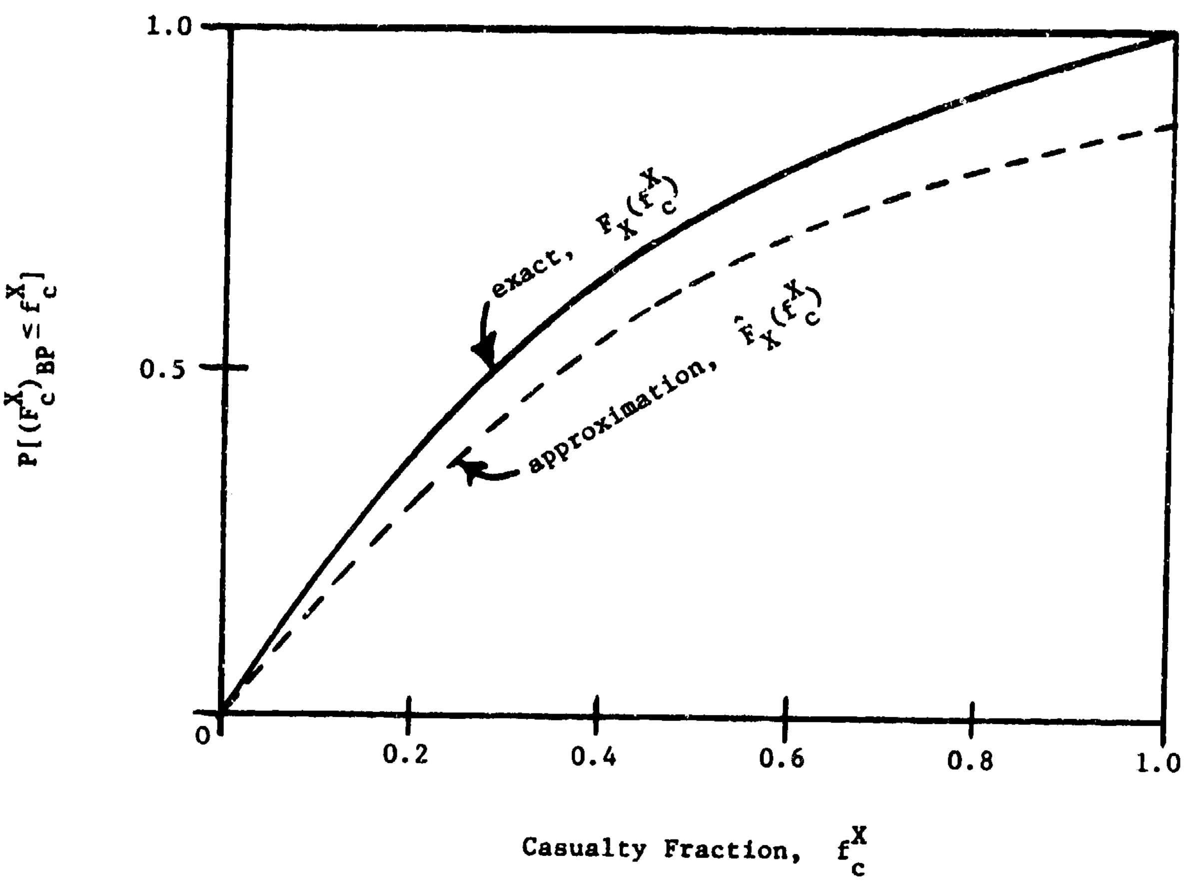


Figure 3.10. Exact and approximate theoretical distribution functions for X's casualty-fraction breakpoint for $\lambda_X = 2.0$ in the exponential case in which $F_X(s)$ is given by (3.8.17).

and (3.8.22) for $\lambda_X, \lambda_Y \geq 5.0$. Such approximations have been freely used by H. K. WEISS [25] in his very original and significant examination of combat data from the U. S. Civil War.[19]

Thus, we will make the approximations (3.8.21) and (3.8.22), and then we can invoke the general results of Table 3.II to obtain the approximate results shown in Table 3.V. For $\lambda_X, \lambda_Y \geq 5.0$, the approximate results should be very close to the exact ones (and almost indistinguishable for $\lambda_X, \lambda_Y \geq 10.0$). Figure 3.11 contrasts how the probability that Y wins depends on the "normalized" initial force ratio $\gamma = (a/b)(y_0/x_0)$ for such exponentially distributed breakpoints with how it does for uniformly distributed ones. In this figure we also see the influence of the ratio λ_X/λ_Y on the probability that Y wins. For the exponentially-distributed breakpoints, the approximate probability of a Y win $\hat{P}_Y$, is given by (3.8.20) for the curves shown in Figure 3.11. Finally, Figure 3.12 shows some theoretical casualty-fraction distributions computed according to exact results [10]. In this figure $P(f_z < u|W_z)$ denotes $P[(F_c^Z)_f \leq u|Z \text{ wins}]$ for $Z = X, Y$. We observe in (a) of Figure 3.12 that $P[(F_c^X)_f \leq u|X \text{ wins}]$ $= P[(F_c^Y)_f \leq u|Y \text{ wins}] = P[(F_c^X)_f \leq u|Y \text{ wins}] = P[(F_c^Y)_f \leq u|X \text{ wins}]$, i.e. for $\gamma = 1$ the casualty-fraction distribution is the same for both X and Y, regardless of who wins (see HELMBOLD [10, p. 84]). This is not true for $\gamma \neq 1$, and (b) and (c) of Figure 3.12 show how X's casualty-fraction distribution depends on who wins.

Thus, for exponentially distributed breakpoints the parameters dependence of battle outcome on model parameters is most easily seen by considering approximate results such as those given in Table 3.V. Although these approximations are only "good" for $\lambda_X, \lambda_Y \geq 5.0$, they do afford a quick look at the general relationship between battle outcome and model parameters and are to be preferred because of the complexity of the exact

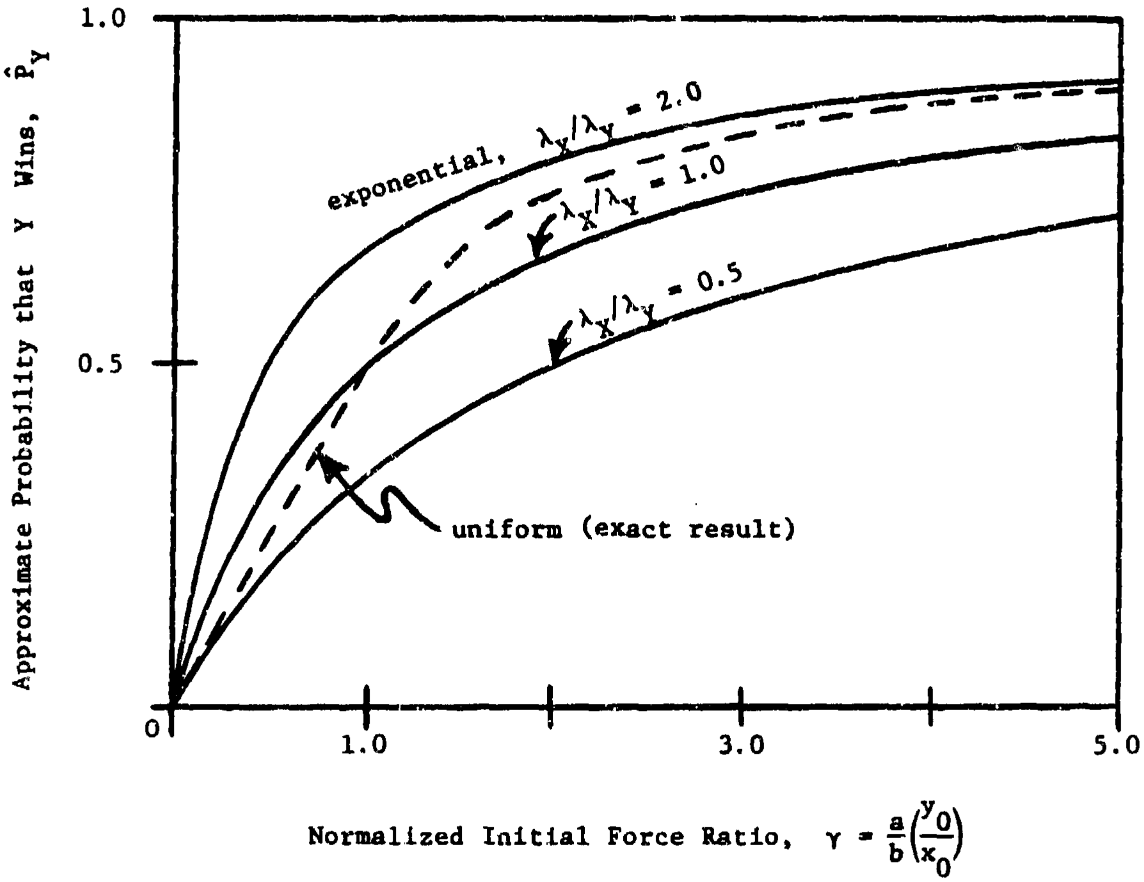
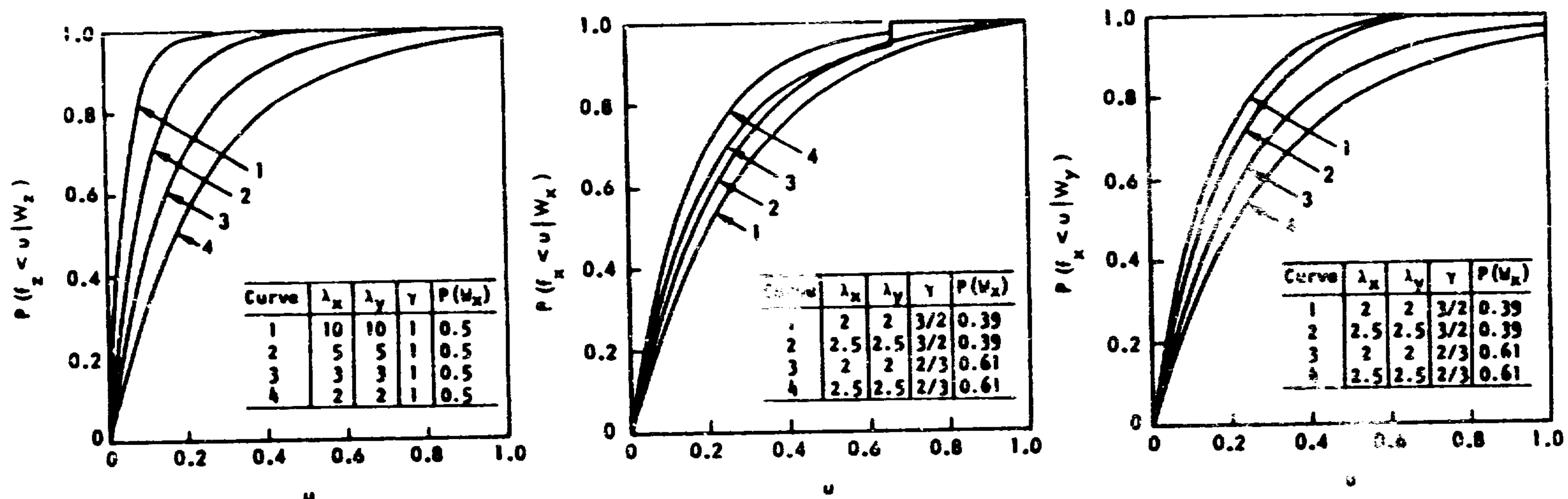


Figure 3.11. The approximate probability $\hat{P}_Y$ that Y wins as a function of the normalized initial force ratio $\gamma = \frac{a}{b}\left(\frac{y_0}{x_0}\right)$ for battle with deterministic FT|FT attrition and random breakpoints. For uniformly distributed breakpoints, the probability that Y wins (dashed line) is exact.



(a) Given a win by either side

(b) Given that the attacker wins

(c) Given that the defender wins

Figure 3.12. Theoretical casualty-fraction distribution for battle with deterministic FT|FT attrition and exponentially distributed breakpoints (from HELMBOLD [10]).

TABLE 3.V. Approximate Results for Battle with Deterministic FT|FT Attrition and Exponentially Distributed Casualty-Fraction Breakpoints with λ_X, $\lambda_Y \geq 5.0$.

Approximate Casualty-Fraction-Breakpoint Distributions:

$$\hat{F}_X(s) = P[(F_c^X)_{BP} \leq s] = 1 - e^{-\lambda_X s}, \qquad \hat{F}_Y(t) = 1 - e^{-\lambda_Y t},$$

where

$$s = \gamma t \qquad \text{for all} \quad t \geq 0 .$$

Approximate Probability of Winning:

$$\hat{P}_Y = \frac{(\lambda_X/\lambda_Y)}{[(\lambda_X/\lambda_Y) + 1/\gamma]}$$

Approximate Casualty-Fractional Distributions:

$$\hat{P}[(F_c^Y)_f \leq q|X \text{ wins}] = \hat{P}[(F_c^Y)_f \leq q|Y \text{ wins}]$$
$$= \hat{P}[(F_c^Y)_f \leq q] = 1 - e^{-(\gamma\lambda_X+\lambda_Y)q}$$

Approximate Average Casualty Fractions:

$$\hat{\bar{f}}_c^X = \gamma\hat{\bar{f}}_c^Y = \frac{1}{(\lambda_X + \lambda_Y/\gamma)}$$

results (see HELMBOLD [10, pp. 78-82]). For λ_X or $\lambda_Y \leq 3.0$, the exact results are to be preferred. We note, however, that (for example) X's average breakpoint is given by (3.8.18) so that $\lambda_X = 3.0$ corresponds to a unit that would on the average fight to a fractional loss of 0.281 before breaking off the engagement (see Table 3.IV).

Finally, from considering the results given in this section, we see that although it is a very simple model and probably oversimplified, the model with both deterministic FT|FT attrition and deterministic breakpoints (see Section 2.8) does provide a very convenient frame of reference for studying more complex models. For this reason, we have emphasized LANCHESTER's classic combat formulations in Chapter 2.

3.9. Battle-Outcome-Prediction Conditions for Deterministic $F|F$ Attrition Process with Stochastic Breakpoints.

In this section we develop battle-outcome-prediction conditions for LANCHESTER's (deterministic) equations of modern warfare (2.2.1), i.e. the equations for an $F|F$ attrition process, with random breakpoints. As above, we will assume that the two stochastic battle-termination processes are independent, i.e. each side's breakpoint is independent of that for the other side. Results are then given for the two specific casualty-fraction-breakpoint distributions considered above: namely,

(D1) uniformly distributed breakpoints,

and (D2) exponentially distributed breakpoints.

Results have not been as completely developed as and are far more complicated than those above for $FT|FT$ attrition, and we will consequently focus on the probability of winning. Because of the complexity of exact analytical results, a couple of very useful approximations will be considered.

As above, we will begin by developing some general results for $F|F$ attrition. First, let us observe that the state equation (2.2.5) for the $F|F$ attrition process (2.2.1) may be expressed in terms of the casualty fractions f_c^X and f_c^X as

$$f_c^X = 1 - \sqrt{1 + \frac{a}{b}\left(\frac{y_0}{x_0}\right)^2 \{(1-f_c^Y)^2 - 1\}} \quad , \qquad (3.9.1)$$

whence the φ function such that $f_c^X = \varphi(f_c^Y)$ is given by

$$\varphi(t) = 1 - \sqrt{1 + \mu^2\{(1-t)^2 - 1\}} \quad , \qquad (3.9.2)$$

where

$$\mu = \frac{y_0}{x_0}\sqrt{\frac{a}{b}} \quad . \qquad (3.9.3)$$

HELMBOLD [10, pp. 7-8] refers to μ as a measure of the relative advantage of Y over X. In his empirical investigations of combat models, HELMBOLD [8-10] always takes X to be the attacker and Y to be the defender. He then introduces the defender's "advantage parameter," which he defines as

$$V = \ln(\mu) . \tag{3.9.4}$$

Then V will range from $-\infty$ to $+\infty$ and

$$V \begin{cases} < 0 & \text{if } X \text{ (the attacker) has the advantage,} \\ > 0 & \text{if } Y \text{ (the defender) has the advantage.} \end{cases}$$

This terminology, however, is a little misleading, since (for example) for deterministic breakpoints $V < 0$ does not imply that X will always win. Recalling (3.6.3), we see that it is indeed possible for Y to win a fixed-force-level-breakpoint battle (in finite time) when $V < 0$ in cases when the breakpoints are appreciably different in favor of Y (see Table 2.XI). Consequently, we will not refer to μ as a relative advantage parameter but will call μ the "normalized" initial force ratio.

The modified ψ-function, defined by (3.7.6), is given by

$$\psi(t) = \begin{cases} 1 - \sqrt{1 + \mu^2\{(1-t)^2 - 1\}} & \text{for } 0 \leq t \leq t_U, \\ 1 & \text{for } \quad t_U \leq t , \end{cases} \tag{3.9.5}$$

where

$$t_U = \begin{cases} 1 & \text{for } 0 \leq \mu \leq 1, \\ 1 - \sqrt{1 - 1/\mu^2} & \text{for } 1 \leq \mu. \end{cases} \tag{3.9.6}$$

Also,

$$\psi^{-1}(s) = \begin{cases} 1 - \sqrt{1 + \{(1-s)^2 - 1\}/\mu^2} & \text{for } 0 \leq s \leq s_U, \\ 1 & \text{for } s_U \leq s , \end{cases} \tag{3.9.7}$$

where

$$s_U = \begin{cases} 1 - \sqrt{1 - \mu^2} & \text{for } 0 \leq \mu \leq 1, \\ 1 & \text{for } 1 \leq \mu. \end{cases} \tag{3.9.8}$$

The key general battle-outcome prediction expressions for an $F|F$ attrition process may be obtained by combining the above results with the general expressions given in Table 3.I. For example, the probability that Y will win is given by

$$P_Y = \begin{cases} \int_0^{s_U} \bar{F}_Y(\psi^{-1}(s))\, dF_X(s) & \text{for } 0 \leq \mu \leq 1, \\ \int_0^{1} \bar{F}_Y(\psi^{-1}(s))\, dF_X(s) & \text{for } 1 \leq \mu, \end{cases} \tag{3.9.9}$$

where $\psi^{-1}(s)$ is given by (3.9.7). Again (recall (3.8.6)), we observe that the upper limit of integration depends on the "normalized" initial force ratio, $\mu = (y_0/x_0)\ \sqrt{a/b}$. Let us recall (see (2.8.3) or Proposition 3.6.1) that for determinisitic breakpoints Y will win a fixed-force-level

breakpoint battle (in finite time) if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}\left\{\frac{1-(f_{BP}^Y)^2}{1-(f_{BP}^X)^2}\right\}} \quad . \tag{3.9.10}$$

Hence, for equal breakpoints, i.e. $f_{BP}^X = f_{BP}^Y$, Y will win if and only if $\mu > 1$. Thus, we see (not unexpectedly) that results for random breakpoints are closely related to those for deterministic ones. The simple, totally deterministic model (2.8.12) provides a very important frame of reference for examining more complex models with random effects.

We can already see that battle-outcome-prediction results for $F|F$ attrition with random breakpoints will be considerably more complex than those for the $FT|FT$ case and (quite possibly, although we cannot prove this assertion) not expressible in terms of so-called "elementary" functions. Hence, some type of simplifying approximation is desirable. HELMBOLD [10, p. 48] has suggested linearizing $\psi(t)$. In other words, if we expand $\varphi(t)$, as given by (3.9.2), in a TAYLOR series about $t = 0$, then we obtain

$$\varphi(t) = \mu^2 t + O(t^2) \quad , \tag{3.9.11}$$

where $O(t^2)$ denotes terms that are of the same order of magnitude as t^2 for t "small." Ignoring the higher order terms, we obtain HELMBOLD's approximation for $\varphi(t)$, namely

$$\hat{\varphi}(t) = \mu^2 t \quad , \tag{3.9.12}$$

whence

$$\psi(t) = \begin{cases} \mu^2 t & \text{for } 0 \leq t \leq 1/\mu^2 , \\ 1 & \text{for } 1/\mu^2 \leq t , \end{cases} \tag{3.9.13}$$

and

$$\hat{\psi}^{-1}(s) = \begin{cases} s/\mu^2 & \text{for } 0 \leq s \leq \mu^2 , \\ 1 & \text{for } \mu^2 \leq s . \end{cases} \tag{3.9.14}$$

Hence, we can invoke all the results for the $FT|FT$ process (see, for example, (3.8.6) through (3.8.10)) with $\gamma = \mu^2$ to obtain approximate results for the $F|F$ attrition process. For example,

$$\hat{P}_Y = \begin{cases} \int_0^{\mu^2} \bar{F}_Y\left(\frac{s}{\mu^2}\right) dF_X(s) & \text{for } 0 \leq \mu \leq 1 , \\ \int_0^{1} \bar{F}_Y\left(\frac{s}{\mu^2}\right) dF_X(s) & \text{for } 1 \leq \mu , \end{cases} \tag{3.9.15}$$

where $\hat{P}_Y$ denotes an approximate probability that Y will win.

As we have done in the previous section, let us now consider two specific breakpoint distributions. First, we consider uniformly distributed casualty-fraction breakpoints, i.e. we assume that (3.8.11) holds. For simplicity, let us focus on the probability of winning, say (for example) for Y. In this case, (3.8.11) and (3.9.9) yield

$$P_Y = \begin{cases} \int_0^{s_U} \{1 - \psi^{-1}(s)\}ds & \text{for } 0 \le \mu \le 1 , \\ \int_0^{1} \{1 - \psi^{-1}(s)\}ds & \text{for } \quad 1 \le \mu, \end{cases} \tag{3.9.16}$$

where $\psi^{-1}(s)$ is given by (3.9.7) and s_U is given by (3.9.8). Fortunately, the integral in (3.9.16) may be evaluated in terms of elementary functions, namely

$$P_Y = \frac{1}{2} + \left(\frac{\mu^2-1}{4\mu}\right) \ln \left\{\left|\frac{\mu+1}{\mu-1}\right|\right\} , \tag{3.9.17}$$

which is the exact result for Y's win probability for a battle with F|F attrition and uniformly distributed casualty-fraction breakpoints. The dependence of this probability on the normalized initial force ratio, $\mu = (y_0/x_0)\sqrt{a/b}$, is shown in Figure 3.13.

If we compare the shape of the plot of P_Y versus the normalized initial force ratio in Figure 3.5 with that in Figure 3.13 (bearing in mind, however, that $\gamma \neq \mu$), we see that the curve is much steeper in the neighborhood of $\mu = 1$ in Figure 3.13 than it is near $\gamma = 1$ in Figure 3.5. This is a reflection of the fact that additional initial forces have a much greater impact on the outcome of a battle with F|F attrition than one with FT|FT attrition (recall Section 2.9 on concentration of forces).

Now let us consider HELMBOLD's approximation of using a linearization of the nonlinear ψ-function, denoted as $\hat{\psi}(t)$ [see (3.9.13) and (3.9.14)] Again, we will focus on the probability that Y will win. In this case,

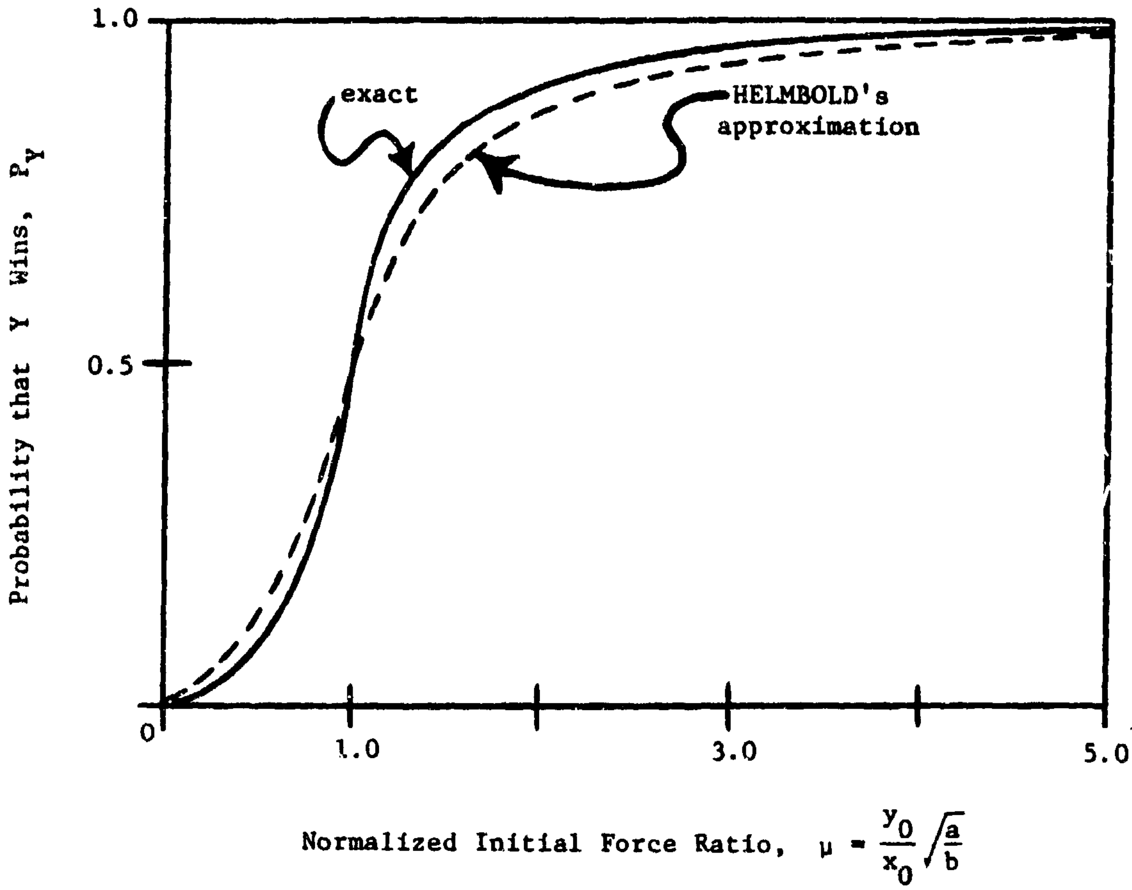


Figure 3.13. Relationship between the normalized initial force ratio $\mu = \frac{y_0}{x_0}\sqrt{\frac{a}{b}}$ and the probability of winning for battle with deterministic $F|F$ attrition and uniformly distributed breakpoints. The dashed curve shows the approximate probability based on HELMBOLD's linearization of $\varphi(t)$.

(3.8.11) and (3.9.15) yield

$$\hat{P}_Y = \begin{cases} \mu^2/2 & \text{for } 0 \le \mu \le 1, \\ 1 - 1/(2\mu^2) & \text{for } 1 \le \mu. \end{cases} \tag{3.9.18}$$

This approximate win probability is shown in Figure 3.13 as the dashed line. We see that for uniformly distributed breakpoints, HELMBOLD's approximation is quite good. A complete error analysis of HELMBOLD's approximation is, however, beyond the scope of our current examination. Other results (e.g. various casualty-fraction distributions) may be obtained in a similar fashion. Some additional results are to be found in HELMBOLD [10].

Finally, let us briefly consider the case of <u>exponentially distributed casualty-fraction breakpoints</u>, i.e. we assume that (3.8.17) holds. Exact results in this case are difficult to obtain so that some type of approximation seems in order. As above, let us focus on the probability that Y will win. If we use WEISS's approximations (3.8.21) and (3.8.22) for the breakpoint distributions and HELMBOLD's linearization (3.9.12) of the φ-function, then for $\lambda_X, \lambda_Y \ge 5.0$ the following should be a good approximation for P_Y

$$\hat{P}_Y = \frac{(\lambda_X/\lambda_Y)}{[(\lambda_X/\lambda_Y) + 1/\mu^2]} . \tag{3.9.19}$$

Other results may be obtained in a similar fashion.

*3.10. Another Model that Considers Unit Deterioration Due to Attrition.

Most ground-combat models determine the level of combat effectiveness of forces by considering the loss of personnel or supplies and equipment. For example, ATLAS (see [7, p. 6-3]) considers that the effectiveness (measured in terms of a firepower score) of a combat unit to depend on the percent casualties of the unit, the level of the unit's supplies and equipment, and the tactical posture of the unit, i.e. whether it is attacking or defending. In particular, (nonlinear) effectiveness curves relating percent degradation in unit effectiveness to percent casualties are used in ATLAS (see [7, p. 6-4]). These curves implicitly supply unit breakpoints by providing a casualty level (equivalently, a force level) at which a unit ceases to be effective and must break off the engagement. Accordingly, a major combat modelling issue is to determine how to relate force effectiveness to personnel strength. We will now analytically examine this via LANCHESTER-type models of combat.

Let us consider two homogeneous forces in LANCHESTER-type combat. For illustrative purposes we will model the basic combat attrition process with LANCHESTER's equations for modern warfare (2.2.1), i.e. the equations for an $F|F$ attrition process (see Figure 2.14), although our approach does apply to other attrition structures. We will additionally assume that our Breakpoint Hypothesis holds (see Section 3.2). In this case, we may consider that a force is effective only when its personnel strength is above its breakpoint force level, since the disengagement process is triggered when the unit's breakpoint is reached. Therefore, as first noted in Section 2.8, we should in this case write LANCHESTER's equations for modern warfare as

$$
\begin{cases}
\dfrac{dx}{dt} = \begin{cases} -ay & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise,} \end{cases} \\[2ex]
\dfrac{dy}{dt} = \begin{cases} -bx & \text{for } x > x_{BP} \text{ and } y > y_{BP}, \\ 0 & \text{otherwise.} \end{cases}
\end{cases}
\tag{3.10.1}
$$

It is instructive to examine the (casualty) effectiveness, for example, of the Y force in the above combat model. Measured in terms of its kill rate, the Y-force effectiveness is given by

$$
\begin{pmatrix} \text{Y-force casualty} \\ \text{effectiveness} \end{pmatrix} = \left(-\frac{dx}{dy}\right) = \begin{cases} ay & \text{for } y_{BP} < y \leq y_0, \\ 0 & \text{for } 0 \leq y \leq y_{BP}. \end{cases}
\tag{3.10.2}
$$

This dependence of unit effectivenss on personnel strength (for cases of no replacements and withdrawals, personnel casualties) is diagrammatically shown in Figure 3.14.

We see from Figure 3.14 that this combat formulation suffers from having a discontinuity in force effectiveness when a side reaches its breakpoint: just above its breakpoint a force may be quite effective in producing enemy casualties; while upon reaching its breakpoint, it becomes totally ineffective. This somewhat unsatisfactory situation is the direct consequence of combining the Breakpoint Hypothesis (see Section 3.2) with equations for $F|F$ attrition without any modification of the latter. Thus, this battle-termination model may be considered to be slightly imcompatible with the usual $F|F$ combat dynamics. Moreover, a combat model such as ATLAS [7, Figure 6-4 on p. 6-4] uses a continuous degradation in unit effectiveness (over the strictly linear reduction expected from

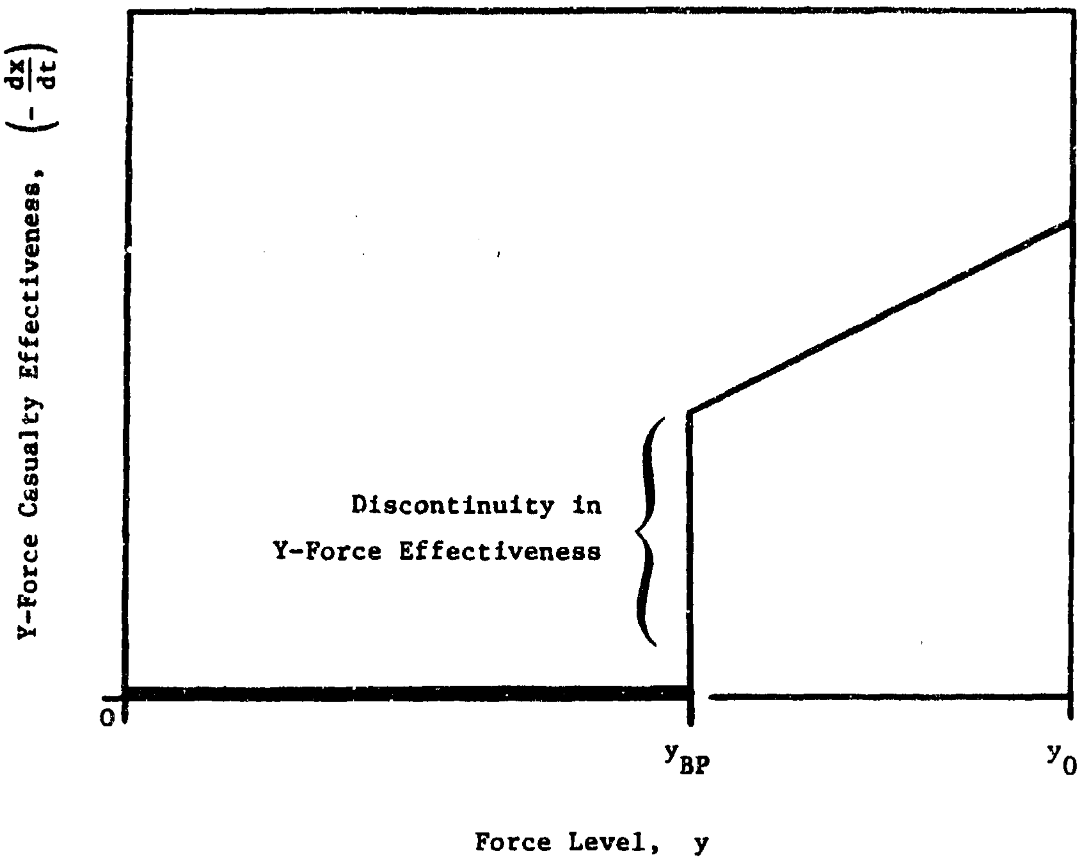


Figure 3.14. Relation between force effectiveness and unit strength for Y force in combat modelled by F|F attrition equations (3.10.1).

reduced force levels) as force levels are reduced through attrition until its breakpoint is reached. Let us therefore develop from physically motivated hypotheses an alternative model that possesses such a feature of unit deterioration.

It seems reasonable to hypothesize that the fraction of a force that is effective depends on the number of casualties that the force has suffered (for cases of no replacements and withdrawals, equivalently, the force level). For example, the loss of one or two men should have little effect except for reducing the unit's force level (i.e. number of available firers). Higher levels of casualties, however, might well affect the organizational integrity of the unit and reduce its effectiveness more than in just direct proportion to its casualty level. SPRING and MILLER [17] and others have postulated such a relationship between the fraction of a force that is effective and the force's casualty level. Let us now consider how such a hypothesis leads to a modification of LANCHESTER's classic equations for modern warfare. We will see that such a hypothesized relationship generally leads to the following type of combat model

$$\left\{\begin{array}{l} \dfrac{dx}{dt} = \begin{cases} -a \cdot f_E^Y\left(\dfrac{y}{y_0}; \dfrac{y_{BP}}{y_0}, \theta_Y\right) \cdot y & \text{for } x_{BP} < x \leq x_0 \text{ and } y_{BP} < y \leq y_0, \\ 0 & \text{otherwise,} \end{cases} \\ \\ \dfrac{dy}{dx} = \begin{cases} -b \cdot f_E^X\left(\dfrac{x}{x_0}; \dfrac{x_{BP}}{x_0}, \theta_X\right) \cdot x & \text{for } x_{BP} < x \leq x_0 \text{ and } y_{BP} < y \leq y_0, \\ 0 & \text{otherwise,} \end{cases} \end{array}\right. \tag{3.10.3}$$

where f_E^Y denotes the fraction of the X force that is effective and depends on y/y_0 and the parameters y_{BP}/y_0 and θ_Y. Similarly for f_E^X. Here θ_Y denotes all other model parameters that pertain to Y. If we express, for example, y_{BP} in the form given by (3.2.2); then $y_{BP}/y_0 = f_{BP}$ and we then have

$$f_E^Y = f_E^Y(y/y_0;\ f_{BP}^Y,\theta_Y) \ . \tag{3.10.4}$$

We will develop below that SPRING and MILLER's [17] functional relationship between effectiveness and casualties yields

$$f_E^Y(y/y_0;f_{BP}^Y,f_I^Y,\nu) = (1-f_I^Y)\ \left\{1 - \left(\frac{y_0-y}{y_0-y_{BP}}\right)^{\nu}\right\}, \tag{3.10.5}$$

where f_I^Y denotes the fraction of the initial Y force that is inherently ineffective in combat (i.e. they never do fire their weapons) and y_{BP} is given by the analogue of (3.2.2). Similarly for $f_E^X(x/x_0;f_{BP}^X,f_I^X,\mu)$.

We will now develop the expression for f_E^Y as given by (3.10.5). Let f_I^Y now denote the fraction of the surviving Y force that is ineffective and recall that Y's casualty fraction is given by

$$f_c^Y = \frac{y_0-y}{y_0} \ . \tag{3.10.6}$$

We will denote Y's casualty fraction at his breakpoint when $y = y_{BP}$ as $(f_c^Y)_{BP}$. SPRING and MILLER [17, pp 12-17] have postulated a relationship such as that shown in Figure 3.15 between the fraction of survivors that are ineffective, f_I^Y, and the casualty fraction, f_c^Y. The shape of the curve in Figure 3.15 suggests the following type of functional relation

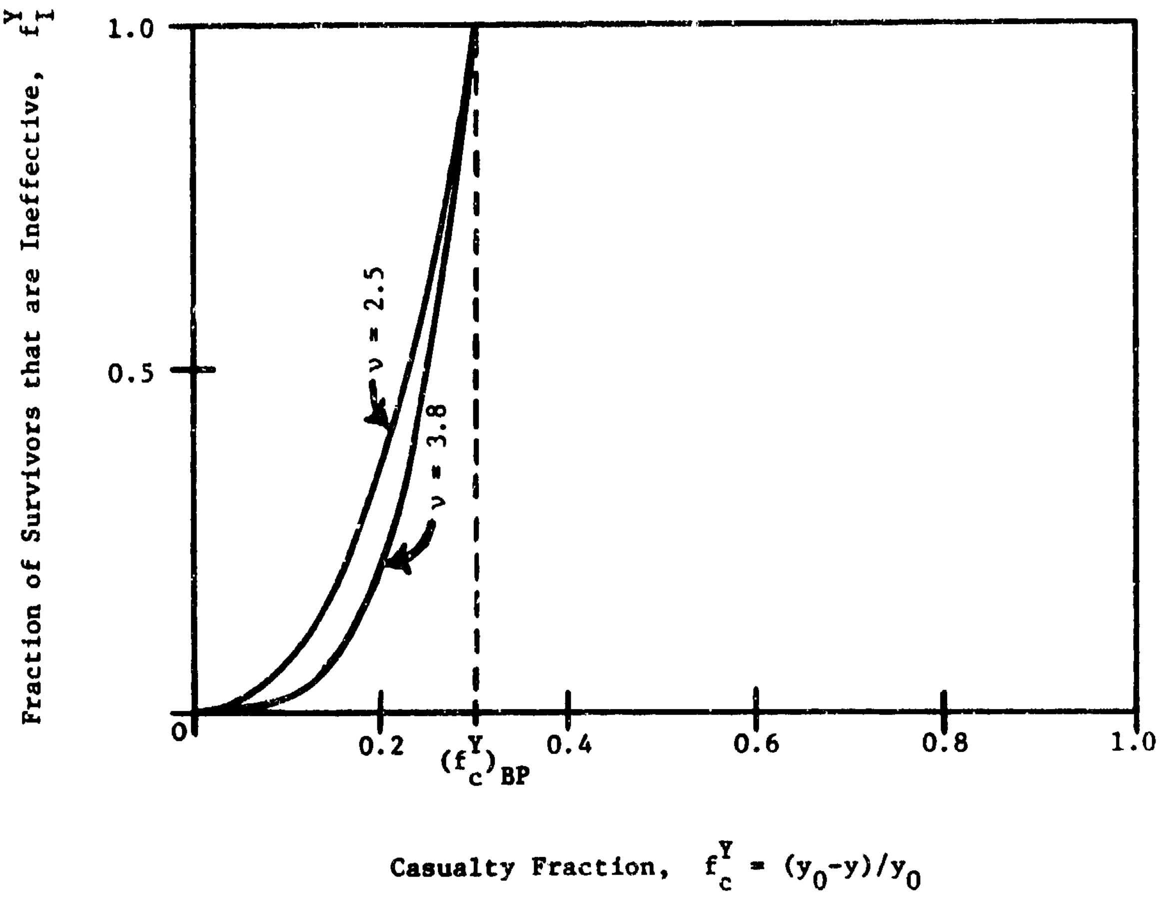


Figure 3.15. Functional relation between a unit's casualty fraction, f_c^Y, and the fraction of the surviving force that is ineffective, f_I^Y, as originally postulated by SPRING and MILLER [17] for an infantry company in the attack.

$$f_I^Y = f_I^Y(f_c^Y) = C_1 + C_2(f_c^Y)^\nu , \qquad (3.10.7)$$

where C_1 and C_2 are constants. If $f_I^Y(0) = 0$ and $f_I^Y[(f_c^Y)_{BP}] = 1$, then $C_1 = 0$ and $C_2 = 1/[(f_c^Y)_{BP}]$ so that (3.10.7) becomes

$$f_I^Y = \left(\frac{y_0 - y}{y_0 - y_{BP}}\right)^\nu . \qquad (3.10.8)$$

REDDOCK [15] has found that values of ν between 2.5 and 3.8 give a reasonable fit to the curves in SPRING and MILLER [17].

We will, however, modify the type of functional relation originally considered by SPRING and MILLER [17] by assuming that a certain fraction of the Y force, denoted as $(f_I^Y)_0$, will essentially always be ineffective and will never fire their weapons in combat, regardless of what the casualty level is.[20] Assuming that the remaining force suffers degradation as postulated by SPRING and MILLER, we will consider the type of relation shown in Figure 3.16, namely

$$f_I^Y = (f_I^Y)_0 + \{1 - (f_I^Y)_0\}\left(\frac{y_0 - y}{y_0 - y_{BP}}\right)^\nu , \qquad (3.10.9)$$

where $(f_I^Y)_0$ denotes the fraction of the initial Y force that is ineffective (i.e. that never fires its weapons). It should be noted that (3.10.9) reduces to (3.10.8) when $(f_I^Y)_0 = 0$. The fraction of the Y force that is effective, denoted as f_E^Y, is then given by (3.10.5), where for convenience we have denoted $(f_I^Y)_0$ simply as f_I^Y.

Thus, our combat model that considers deterioration in unit fire effectiveness due to casualties and that not every man fires his weapon

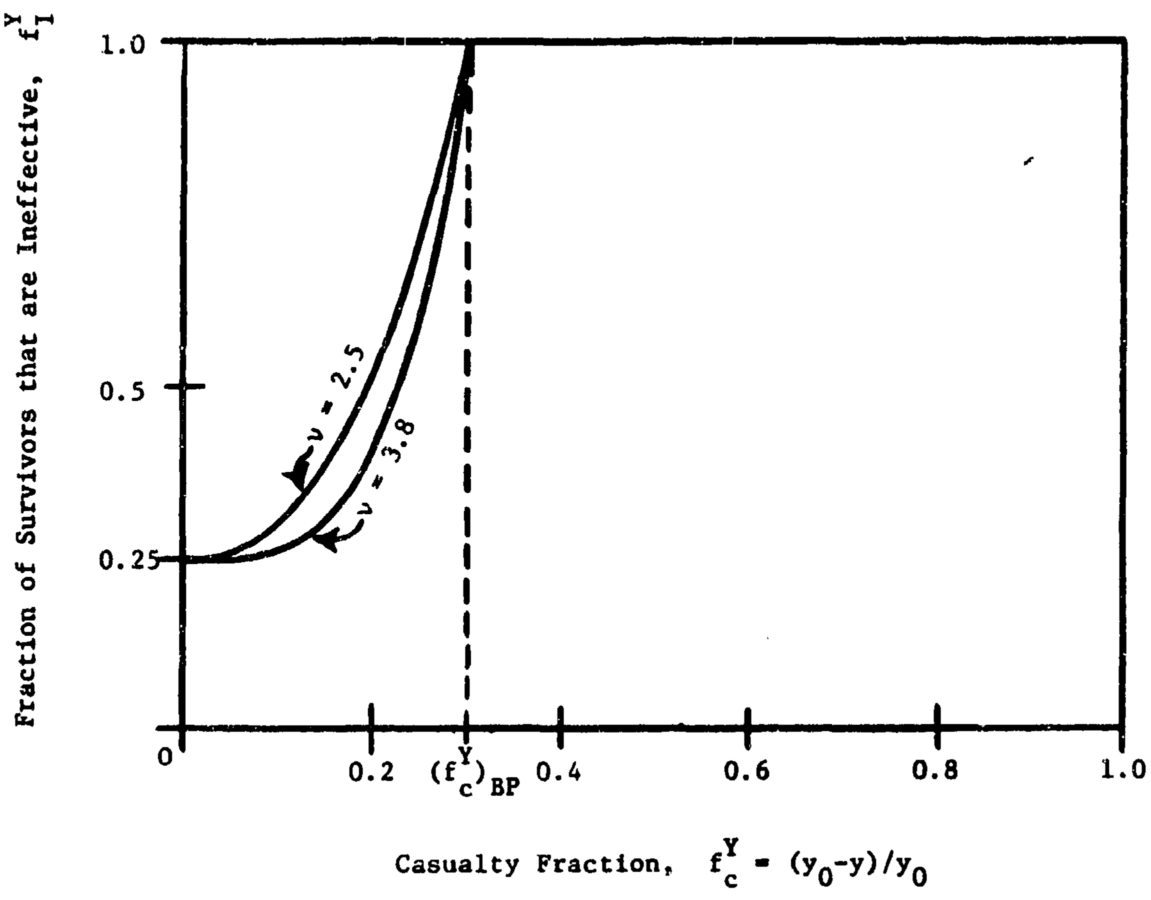


Figure 3.16. Functional relation between a unit's casualty fraction, f_c^Y, and the fraction of the surviving force that is ineffective, f_I^Y, for an infantry company in the attack with 0.25 of the initial force always ineffective.

in combat may be written as

$$\begin{cases} \dfrac{dx}{dt} = \begin{cases} -a(1-f_I^Y)\left\{1 - \left(\dfrac{y_0-y}{y_0-y_{BP}}\right)^{\nu}\right\} y & \text{for } x_{BP} < x \le x_0 \text{ and } y_{BP} < y \le y_0, \\ 0 & \text{otherwise,} \end{cases} \\ \\ \dfrac{dy}{dt} = \begin{cases} -b(1-f_I^X)\left\{1 - \left(\dfrac{x_0-x}{x_0-x_{BP}}\right)^{\mu}\right\} x & \text{for } x_{BP} < x \le x_0 \text{ and } y_{BP} < y \le y_0, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{3.10.10}$$

where, for example, f_I^Y denotes the fraction of the Y force that is always ineffective. Before proceeding further, let us make a few observations about our modification of LANCHESTER's classic $F|F$ attrition model to incorporate ineffective combatants and unit deterioriation due to attrition. For $f_I^X = f_I^Y = 0$ and $\mu = \nu = +\infty$, the equations (3.10.10) reduce to (3.10.1). However, our combat formulation (3.10.10) is very nonlinear in the force levels. In Figure 3.17 we show how the Y-force effectiveness for our new model (3.10.10) varies with the Y force level. There is no longer a discontinuity in unit effectiveness at the unit's breakpoint. Moreover, it is indeed surprising (as we will show below) that when $f_{BP}^X = f_{BP}^Y$, $f_I^X = F_I^Y$, and $\mu = \nu$, the Y force will still win (in finite time) if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}} \quad . \tag{3.10.1}$$

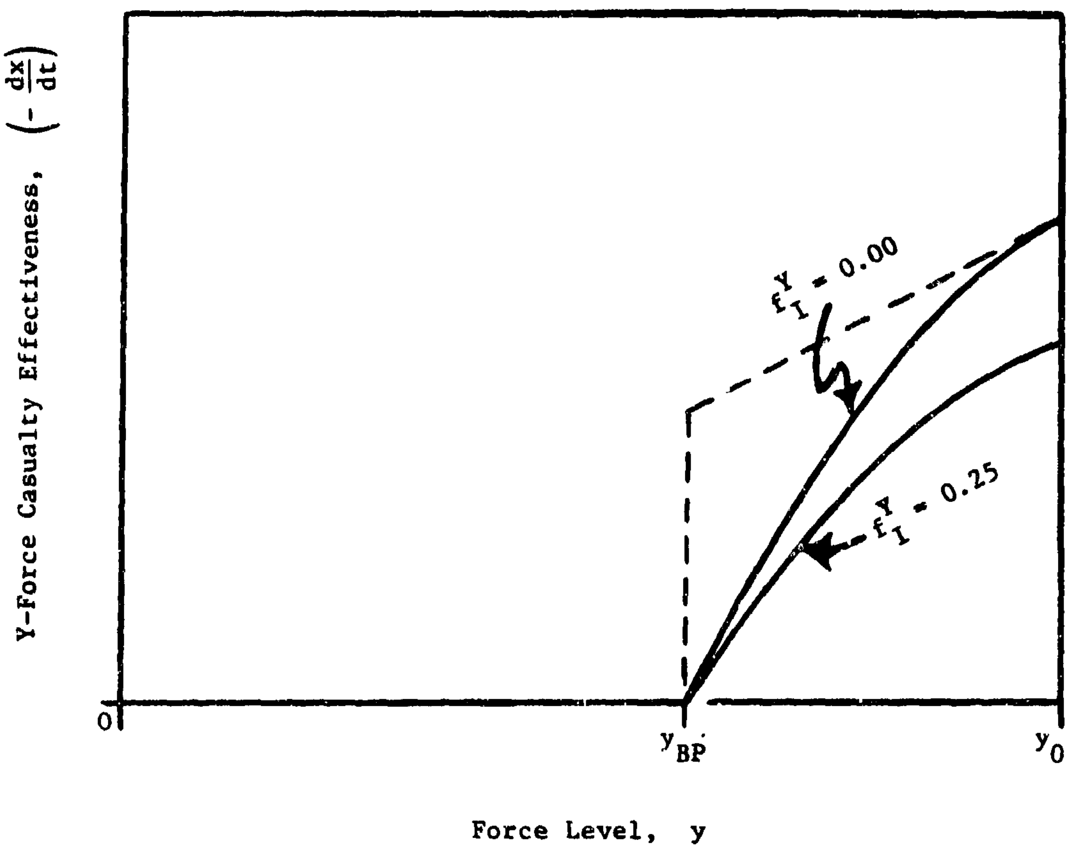


Figure 3.17. Relation between force effectiveness and unit strength for Y force in combat modelled by nonlinear equations (3.10.10). [Here we have let $(f_c^Y)_{BP} = 0.4$ and $\nu = 2.5$. The dashed line is for combat modelled by the $F|F$ attrition equations (3.10.1) (see Figure 3.14).]

To obtain the state equation that relates the X and Y force levels for $x \geq x_{BP}$ and $y \geq y_{BP}$, we first divide the first equation of (3.10.10) by the second to obtain the instantaneous (or differential) casualty-exchange ratio

$$\frac{dx}{dy} = \frac{a(1-f_I^Y)\left\{1 - \left(\frac{y_0-y}{y_0-y_{BP}}\right)^{\nu}\right\} y}{b(1-f_I^X)\left\{1 - \left(\frac{x_0-x}{x_0-x_{BP}}\right)^{\mu}\right\} x} . \tag{3.10.12}$$

Separating variables and integrating, we obtain the following state equation for $x \geq x_{BP}$ and $y \geq y_{BP}$

$$H(x_0,x) = K(y_0,y) , \tag{3.10.13}$$

where

$$H(x_0,x) = \frac{b}{2}(x_0^2-x^2)-b\left(\frac{x_0-x_{BP}}{\mu+1}\right)\left\{\frac{(\mu+1)x+x_0}{\mu+2}\right\}\left(\frac{x_0-x}{x_0-x_{BP}}\right)^{\mu+1} , \tag{3.10.14}$$

and

$$K(y_0,y) = \frac{a}{2}(y_0^2-y^2)-a\left(\frac{y_0-y_{BP}}{\nu+1}\right)\left\{\frac{(\nu+1)y+y_0}{\nu+2}\right\}\left(\frac{y_0-y}{y_0-y_{BP}}\right)^{\nu+1} . \tag{3.10.15}$$

We have not been able to obtain a time solution, e.g. the X force level as a function of time $x(t)$, for the model (3.10.10), even in cases of particular (but finite) values for μ and ν. It does not appear that the time solution, e.g. $x(t)$, is expressible in terms of any of the standard functions of mathematical analysis. We can, however, use

finite-difference methods to "numerically integrate" (3.10.10) and obtain an approximate value of, for example, the X force level, denoted as $\hat{x}(t)$ (see Chapter 7). Such a numerical solution is usually generated (with the help of a digital computer) for particular numerical values of the model parameters and initial conditions, and consequently it does not by itself provide any insights into the dynamics of combat. Moreover, it is even essentially impossible to explicitly solve (3.10.13) through (3.10.15) for x in terms of y (except for the special case of $\mu = \nu = 1$). Thus, the state equation (3.10.13) appears to be of little use. However, we will now show that Y will win (in finite time) if and only if

$$H(x_0, x_{BP}) < K(y_0, y_{BP}) \ . \tag{3.10.16}$$

Furthermore, (3.10.16) is of considerable value for providing some important insights into the dynamics of combat.

In developing the Y-victory-prediction condition (3.10.16), we first observe that equations (3.10.10) are of the same form as (3.3.4), namely

$$\begin{cases} \dfrac{dx}{dt} = -\ f(t)\ F_1(x)\ F_2(y) & \text{with}\ \ x(0) = x_0\ , \\ \dfrac{dy}{dt} = -\ f(t)\ G_1(x)\ G_2(y) & \text{with}\ \ y(0) = y_0\ , \end{cases} \tag{3.10.17}$$

where $f(t) \geq 0$ and F_1, F_2, G_1, and $G_2 > 0$ for $x > x_{BP}$ and $y > y_{BP}$. Other assumptions will be stated presently. We have shown (see Section 3.3) that Y will win (in finite time) if and only if $x_{BP} > g(y_{BP})$, where

$$g(y) = \rho^{-1}[p(x_0) - q(y_0) + q(y)] \ , \qquad (3.10.18)$$

$p^{-1}(\eta)$ is a strictly increasing function of η, and p and q are given by (3.3.6) and (3.3.7), respectively. Thus Y will win if and only if

$$x_{BP} > p^{-1}[p(x_0) - q(y_0) + q(y_{BP})] \ ,$$

or

$$p(x_{BP}) > p(x_0) - q(y_0) + q(y_{BP}) \ , \qquad (3.10.19)$$

since $p(x)$ is strictly increasing. Recalling (3.3.6) and (3.3.7), we see that (3.10.16) follows from (3.10.19). Let us observe that $p(x) = p(x;x_0,x_{BP},\theta_X)$, where θ_X denotes the X-force parameters.

It remains to show that t_{BP}^X is finite. We now make the following assumptions:

(1) F_1, F_2, G_1, and G_2 are strictly increasing functions of their arguments with $F_1(x_{BP}) > 0$,

(2) $\int_0^T f(t)dt$ exists and is finite for any finite T but $\lim_{T\to+\infty} \int_0^T f(t)dt = +\infty$.

Then (3.10.16) implies that t_{BP}^X is finite. The proof is as follows. If $H(x_0,x_{BP}) < K(y_0,y_{BP})$, then $y \geq y_f > y_{BP}$ so that

$$\frac{dx}{dt} = - f(t)\ F_1(x)\ F_2(y) \leq - f(t)\ F_1(x_{BP})\ F_2(y_f) \ ,$$

whence

$$x(t) \leq x_0 - F_1(x_{BP})\, F_2(y_f) \int_0^t f(s)\, ds \;.$$

From the assumption that $\lim_{T\to+\infty} \int_0^T f(t)dt = +\infty$, it follows that $x(t) \to x_{BP}$ in finite time. Thus, we have proven the following important proposition.

PROPOSITION 3.10.1: Consider the LANCHESTER-type equations (3.10.17) and assume that:

(A1) F_1, F_2, G_1, and $G_2 > 0$ for $x > x_{BP}$ and $y > y_{BP}$,

(A2) $F_1(x)$, $F_2(y)$, $G_1(x)$, and $G_2(y)$ are strictly increasing functions of their arguments for $x \geq x_{BP}$ and $y \geq y_{BP}$ with $F_1(x_{BP}) > 0$,

(A3) $f(t) \geq 0$, $\int_0^T f(t)dt$ exists and is finite for every finite value of T, and $\lim_{T\to+\infty} \int_0^T f(t)dt = +\infty$

Then Y will win a fixed-force-level-breakpoint battle in finite time if and only if

$$H(x_0, x_{BP}) < K(y_0, y_{BP}) \;, \tag{3.10.20}$$

where

$$H(x_0, x) = \int_x^{x_0} \frac{G_1(\xi)}{F_1(\xi)}\, d\xi \quad \text{and} \quad K(y_0, y) = \int_y^{y_0} \frac{F_2(\xi)}{G_2(\xi)}\, d\xi \;. \tag{3.10.21}$$

Considering (3.10.14), (3.10.15), and (3.10.20), we may develop victory-prediction conditions for our combat model (3.10.10). We state these results as Proposition 3.10.2.

PROPOSITION 3.10.2: Consider combat modelled by (3.10.10). Y will win a fixed-force-level-breakpoint battle in finite time if and only if

$$\frac{x_0}{y_0} < \sqrt{\frac{a}{b}}\; Q(f^X_{BP}, f^Y_{BP}, f^X_I, f^Y_I, \mu, \nu) \;, \tag{3.10.22}$$

where

$$Q(f^X_{BP}, f^Y_{BP}, f^X_I, f^Y_I, \mu, \nu)$$

$$= \sqrt{\left(\frac{1-f^Y_I}{1-f^X_I}\right)\left(\frac{1-f^Y_{BP}}{1-f^X_{BP}}\right)\left\{\frac{(\nu+3)/(\nu+1)+f^Y_{BP}}{(\mu+3)/(\mu+1)+f^X_{BP}}\right\}\left(\frac{1+2/\mu}{1+2/\nu}\right)} \tag{3.10.23}$$

It may be shown that the function $Q(f^X_{BP}, f^Y_{BP}, f^X_I, f^Y_I, \mu, \nu)$ possesses the following properties for $0 \le f^X_{BP}, f^Y_{BP}, f^X_I, f^Y_I < 1$ and $\mu, \nu > 0$:

(P1) $\dfrac{\partial Q}{\partial f^X_{BP}}, \dfrac{\partial Q}{\partial f^X_I} > 0$,

(P2) $\dfrac{\partial Q}{\partial f^Y_{BP}}, \dfrac{\partial Q}{\partial f^Y_I} < 0$,

(P3) $\dfrac{\partial Q}{\partial \mu} < 0$,

(P4) $\dfrac{\partial Q}{\partial \nu} > 0$,

and

(P5) $Q(f_{BP}, f_{BP}, f_I, f_I, \mu, \nu) = 1.$

It follows that, for example, if $f_{BP}^Y \geq f_{BP}^X$, $f_I^Y \geq f_I^X$, and $\mu \geq \nu$, then $Q \leq 1$. In this case, X will win (in finite time) if

$$\frac{x_0}{y_0} > \sqrt{\frac{a}{b}} \geq Q\sqrt{\frac{a}{b}}$$

Our analytical model is particularly valuable because it yields a battle-outcome-prediction condition, namely (3.10.22), that explicitly shows the parametric dependence of battle outcome on model parameters. Sensitivity analysis has thus been greatly facilitated. Proposition 3.10.2 is particularly significant because it shows us that the outcome of battle depends on only seven factors (three relative factors and four model parameters), even though our combat model (3.10.10) (with battle-termination conditions included) contains ten independent parameters: namely, a, b, x_0, y_0, f_I^X, f_I^Y, f_{BP}^X, f_{BP}^Y, μ, and ν. Thus, the outcome of a fixed-force-level-breakpoint battle modelled with the combat dynamics (3.10.10) depends on the following seven factors:

(F1) the initial force ratio, $u_0 = x_0/y_0$,

(F2) relative fire effectiveness, $R = a/b$,

(F3) relative fraction of initially effective forces, $\rho_E^0 = (1-f_I^Y)/(1-f_I^X)$,

(F4-5) the two breakpoint-force-level fractions, f_{BP}^X and f_{BP}^Y,

(F6-7) the two unit degradation parameters, μ and ν.

For simplicity, let us rewrite the battle-outcome-prediction condition (3.10.22) of Proposition 3.10.2 as

$$Y \text{ will win (in finite times) if and only if } \frac{x_0}{y_0} < Q\sqrt{\frac{a}{b}} . \quad (3.10.24)$$

Thus, X can only win if the initial force ratio x_0/y_0 exceeds the critical value $Q\sqrt{a/b}$. Numerical values for Q and $Q\sqrt{a/b}$ for various representative values of the parameters f^X_{BP}, f^Y_{BP}, f^X_I, f^Y_I, μ, and ν are shown in Table 3.VI. Parameter values were chosen to be representative of an attack by the X forces against Y. These particular parameter values are similar to those used for the numerical examples shown in Table 2.XI of Section 2.8. Before discussing the contents of Table 3.VI, let us observe that our combat model (3.10.10) may be considered to be a generalization of LANCHESTER's classic F|F attrition model, since (for example) $\{1 - [(y_0-y)/(y_0-y_{BP})]^{\nu}\} \to 1$ as $\nu \to +\infty$ for $y \in (y_{BP},y_0]$ so that for $f^X_I = f^Y_I = 0$ the combat model (3.10.10) $\to$ the classic combat model (3.10.1) as $\mu, \nu \to +\infty$. Additionally, two particular cases of parametric values merit special attention. Specifically, the victory-prediction condition (3.10.24) reduces to previous encountered results in the two special cases:

(C1) $f^X_{BP} = f^Y_{BP}$, $f^X_I = f^Y_I$, and $\mu = \nu$,

and

(C2) $f^X_I = f^Y_I$ and $\mu = \nu = +\infty$.

In the first case (C1) in which $f^X_{BP} = f^Y_{BP}$, $f^X_I = f^Y_I$, and $\mu = \nu$; we obtain from (3.10.23) that $Q = 1$ so that the victory-prediction

TABLE 3.VI. Numerical Values for Q and $Q\sqrt{a/b}$ as the Combat Parameters f_{BP}^{X}, f_{BP}^{Y}, f_{I}^{X}, f_{I}^{Y}, μ, and ν are Varied.

CASE	a/b	ρ_E^0	f_{BP}^X	f_{BP}^Y	μ	ν	Q	$Q\sqrt{a/b}$
1	5.0	1.0	$f_{BP}^X =$	f_{BP}^Y	$\mu =$	ν	1.000	2.236
2	5.0	0.8	0.7	0.7	2.5	3.8	0.938	2.097
3	5.0	1.0	0.7	0.7	2.5	3.8	1.048	2.344
4	5.0	1.0	0.7	0.7	3.8	2.5	0.954	2.133
5	5.0	1.2	0.7	0.7	2.5	3.8	1.148	2.568
6	5.0	1.0	0.7	0.5	2.5	3.8	1.288	2.880
7	5.0	1.0	0.7	0.5	5.0	5.0	1.226	2.741
8	5.0	1.0	0.7	0.5	$+\infty$	$+\infty$	1.213	2.712
9	5.0	1.2	0.7	0.5	2.5	3.8	1.411	3.155
10	5.0	1.2	0.8	0.5	2.5	3.8	1.691	3.781

NOTES:

(1) X is the attacker

(2) $\rho_E^0 = (1-f_I^Y)/(1-f_I^X)$.

condition (3.10.24) reduces to previously encountered results in the two special cases:

(C1) $f_{BP}^{X} = f_{BP}^{Y}$, $f_{I}^{X} = f_{I}^{Y}$, and $\mu = \nu$,

and

(C2) $f_{I}^{X} = f_{I}^{Y}$ and $\mu = \nu = +\infty$.

In the first case (C1) in which $f_{BP}^{X} = f_{BP}^{Y}$, $f_{I}^{X} = f_{I}^{Y}$, and $\mu = \nu$; we obtain from (3.10.23) that $Q = 1$ so that the victory-prediction condition (3.10.24) is identical to the force-annihilation-prediction condition given in Proposition 2.2.1. Thus, we see that results for the simple combat model (2.2.1) are basic for understanding more complex combat models such as the one at hand. In the second case (C2) in which $f_{I}^{X} = f_{I}^{Y}$ and $\mu = \mu = +\infty$, we obtain from (3.10.23) that $Q = \sqrt{\{1 - (f_{BP}^{Y})^2\}/\{1 - (f_{BP}^{X})^2\}}$ so that the victory-prediction condition (3.10.24) for the nonlinear combat model (3.10.10) is the same as that for a battle with $F|F$ attrition (see (2.8.3)).

Let us now return to the contents of Table 3.VI. As stated above, parameter values have been chosen to be representative of an attack by the X forces against Y. As we have noted before, one frequently hears in military circles that a three-to-one force ratio is required for a successful attack against an enemy position. Table 3.VI provides some theoretical justification for this well-known rule-of-thumb. We recall that our model (3.10.10) says that X will win if and only if the initial force ratio x_0/y_0 exceeds the critical threshold value $Q\sqrt{a/b}$.

The values for $Q\sqrt{a/b}$ in the last column of Table 3.VI shows us that relatively minor-looking changes in the combat parameters can change this critical value by two hundred percent or so. We observe

that $Q \geq 1$ yields that $x_0/y_0 < \sqrt{a/b}$ implies that Y will win. In other words, if Y can annihilate X in the classic $F|F$ battle (2.2.1), he will win this one modelled by (3.10.10). Moreover, the contents of Table 3.VI are probably most fruitfully studied by the reader referring back to properties (P1) through (P5) of the Q function, which follow (3.10.23). Let us also note that for the situations considered by Table 3.VI the breakpoint force level (expressed as a fraction of the initial force level) and consequently the effect of a given casualty level is greater for an attacking unit than for a defending one. This is because an attack normally requires rapid movement, good coordination, and high organizational integrity (see [7] for further details).

Let us finally note that the qualitative behavior of the nonlinear combat model (3.10.10) is probably best understood by relating it to that for the linear combat model (3.10.1). Thus, the simple model (3.10.1) (equivalently, (2.2.1)) provides an essential frame of reference for studying more complicated combat models. This fact is the reason why we have spent so much time examining the simple model (2.2.1). Moreover, the nonlinear combat model (3.10.10) is just complicated enough so that we apparently cannot express the time solution (e.g. the X force level as a function of time $x(t)$) in terms of "elementary" functions. Furthermore, the state equation (3.10.13) is so complicated that, for example, for $\nu > 0$ and finite it is essentially impossible (except when $\nu = 1$) to solve for y in terms of x. Nevertheless, we were able to explicitly predict battle outcome in all cases. It was indeed somewhat surprising to obtain a victory-prediction condition of the form (3.10.22), i.e. surprising to obtain a sort of "square law." We will now show that this is a general consequence of combat dynamics of the form (3.10.3).

From (3.10.3) we obtain that Y will win (in finite time) if and only if

$$H_1(x_0,x_{BP}) < K_1(y_0,y_{BP}), \tag{3.10-25}$$

where f_E^X is a strictly increasing function of X and is positive for $x \in (x_{BP},x_0]$, and similarly for f_E^Y. Hence, we have

$$H_1(x_0,x_{BP}) = b \int_{x_{BP}}^{x_0} f_E^X\left(\frac{x}{x_0} ; \frac{x_{BP}}{x_0} , \theta_X\right) x \, dx , \tag{3.10-26}$$

and

$$K_1(y_0,y_{BP}) = a \int_{y_{BP}}^{y_0} f_E^Y\left(\frac{y}{y_0} ; \frac{y_{BP}}{y_0} , \theta_Y\right) y \, dy . \tag{3.10-27}$$

However, an integration by parts yields, for example,

$$H_1(x_0,x_{BP}) = bx_{BP} \int_{x_{BP}}^{x_0} f_E^X\left(\frac{x}{x_0}\right)dx - b \int_{x_{BP}}^{x_0} dx \int_{x_{BP}}^{x} f_E^X\left(\frac{u}{x_0}\right)du ,$$

or

$$H_1(x_0,x_{BP}) = bx_0^2 \, F(f_{BP}^X,\theta_X) , \tag{3.10-28}$$

where $F(f_{BP}^X,\theta_X)$ denotes a function of only f_{BP}^X and the other model parameters θ_X, and similarly for K_1 with associated function $G(f_{BP}^Y,\theta_Y)$. It is readily seen that $F(f_{BP}^X,\theta_X) > 0$ for $x_{BP} < x_0$. Thus, in general for the model (3.10.3)

Y will win (in finite time) if and only if $\frac{x_0}{y_0} < \sqrt{\frac{a}{b} \frac{G(f^Y_{BP}, \theta_Y)}{F(f^X_{BP}, \theta_X)}}$. (3.10.29)

This certainly is an unexpected result. Moreover, it shows how intimately the two combat models (3.10.1) and (3.10.3) are related.

3.11. WEISS's Model of Battle Termination.

H. K. WEISS [24] has considered modelling the ending of a war as a MARKOV process (more precisely, as a continuous-parameter MARKOV chain (see Section 4.2)) and has reported fairly good agreement between his model and available historical war data. Subsequently, in his examination of combat data for the U. S. Civil War, WEISS [25] has also considered modelling battle termination as a MARKOV process: every time that a side sustains a casualty, its commander makes a decision as to whether or not to continue the battle. In other words, the basic idea behind WEISS's model of battle termination is that during a battle (as it progresses and casualties mount on both sides) each side considers only its own observed cumulative fractional loss to the moment of evaluation as the sole criterion for deciding whether or not to continue the battle. When a side has decided not to continue the engagement, it will abandon its mission and will try to break off the engagement. Thus, WEISS's model generates a casualty-fraction-breakpoint distribution for each side.[21]

In other words, WEISS [25] has assumed that a side's own fractional loss (or casualty fraction) is the significant variable governing the battle-termination process. After introducing some necessary notation, we will develop WEISS's model, which yields an exponentially-distributed casualty-fraction breakpoint for each side. In our development here, we will focus on just one of the sides engaged in combat. Let f denote the force's own fractional loss, i.e.

$$f = \text{(the side's own fractional loss)}$$
$$= [\text{(initial force level)} - \text{(current force level)}]/\text{(initial force level)}. \quad (3.11.1)$$

The following <u>assumptions</u> are made <u>for WEISS's model of battle termination</u>:

(A1) a side in combat considers only its own observed cumulative fractional loss (i.e. cumulative casualty fraction) to the moment of evaluation as the sole criterion for deciding whether or not to continue the battle; when the side has decided not to continue the engagement, it will abandon its mission and will try to break off the engagement,

(A2) battle termination in the future is independent of what has happened in the past (i.e. independence of nonoverlapping casualty-fraction intervals),

(A3) $$P\left[\begin{array}{l|l}\text{a side terminates the battle at} & \text{the side has continued to} \\ \text{casualty fraction between } f & \text{fight until casualty} \\ \text{and } (f + \Delta f) & \text{fraction } f\end{array}\right]$$

$$= \lambda(f)\Delta f + O((\Delta t)^2) \ ,$$

Let

$$P[\text{side fights at least until casualty fraction} > f]$$

$$= P[\text{breakpoint} > f]$$

$$= \overline{F}(f) \ , \qquad (3.11.2)$$

where $F(f)$ denotes the d.f. for the side's casualty-fraction breakpoint $(F_c)_{BP}$, i.e. $F(f) = P[(F_c)_{BP} \leq f]$. For notational convenience, let us denote $\overline{F}(t)$ as $\Phi(f)$. Then assumptions (A1) through (A3) and the usual conditional-probability arguments yield

$$P\begin{bmatrix}\text{side fights at least until} \\ \text{casualty fraction} > (f+\Delta f)\end{bmatrix}$$

$$= P\begin{bmatrix}\text{side fights at least until} \\ \text{casualty fraction} > f\end{bmatrix} \cdot P\begin{bmatrix}\text{side does not terminate battle} \\ \text{between } f \text{ and } (f+\Delta f)\end{bmatrix}$$

or

$$\Phi(f + \Delta f) = \Phi(f)\,\{1 - \lambda(f)\Delta f\} + O((\Delta f)^2) ,$$

whence

$$\frac{\Phi(f + \Delta f) - \Phi(f)}{\Delta f} = \lambda(f)\,\Phi(f) + O(\Delta f) .$$

Letting $\Delta f \to 0$, we obtain

$$\frac{d\Phi}{df} = -\lambda(f)\Phi .$$

When $f = 0$, we have $\Phi(0) = 1$, since it is certain that a side (if it does initial the battle) will suffer some casualties, i.e., P[side fights at least until casualty fraction > 0] = 1. Thus, assumption (A1) through (A3) yield the following differential equation for the breakpoint complementary d.f.

$$\frac{d\Phi}{df} = -\lambda(f)\Phi \qquad \text{with} \quad \Phi(0) = 1 . \tag{3.11.3}$$

Separating variables and integrating, we obtain

$$\Phi(f) = \exp\{-\int_0^f \lambda(s)ds\} . \tag{3.11.4}$$

Since $F(f)$ is a distribution function, we must have $F(1) = \lim_{\substack{f \to 1 \\ f \geq 1}} F(f) = F(1 + 0) = 1 = 1 - \bar{F}(1 + 0)$. In other words, we must have $\bar{F}(1 + 0) = 0$. However, if $\lambda(f)$ is bounded for all $f \in [0,1]$, then $\Phi(1) = \exp\{-\int_0^1 \lambda(f)df\} > 0$ so that in order for $\Phi(f)$ to be a complementary d.f. we must somehow define $\Phi(1 + 0)$ to be 0. In any case, assumption (A1) through (A3) are not quite compatible with a casualty-fraction-breakpoint distribution function for a continuously-distributed breakpoint.

One way to obtain a distribution function is to take

$$\text{(Modification 1)} \quad F(f) = \begin{cases} 1 - \exp\{-\int_0^f \lambda(s)ds\} & \text{for } 0 \leq f < 1, \\ 1 & \text{for } f = 1. \end{cases} \tag{3.11.5}$$

HELMBOLD[10] has given battle-outcome-prediction results for such exponentially-distributed breakpoints. However, the casualty-fraction breakpoint is no longer continuously distributed. Moreover, on physical grounds, we must have P[casualty-fraction breakpoint ≤ 1] = 1, since a force cannot continue the battle (with probability one) once it has been annihilated. Another way to obtain a distribution function is to rescale $F(f)$ by the appropriate factor, namely

$$\text{(Modification 2)} \quad F(f) = \frac{1 - \exp\{-\int_0^f \lambda(s)ds\}}{1 - \exp\{-\int_0^1 \lambda(s)ds\}}. \tag{3.11.6}$$

However, the battle termination process is not quite MARKOVIAN in this second case. When $\lambda(f)$ = constant, the above expressions for $F(f)$ simplify to

$$\text{(Modification 1)} \quad F(f) = \begin{cases} 1 - e^{-\lambda f}, & \text{for } 0 \leq f < 1, \\ 1 & \text{for } f = 1, \end{cases} \tag{3.11.7}$$

and

$$\text{(Modification 2)} \quad F(f) = \frac{1 - e^{-\lambda f}}{1 - e^{-\lambda}}. \tag{3.11.8}$$

It essentially does not matter which modified expression (i.e. either (3.11.5) or (3.11.6)) we use when $\lambda(f) \geq 5.0$ for all $f \in [0,1]$, since then there is negligible difference between them and consequently approximately the same battle-outcome-prediction results are to be obtained from each. For example, in the case of a constant battle-termination rate λ, the expressions (3.11.7) and (3.11.8) differ by at most 0.007 near $f = 1$ when $\lambda \geq 5.0$ (see Section 3.8). In such cases, essentially the same battle-outcome-prediction results are obtained for either (3.11.7) or (3.11.8), and it is therefore immaterial as to which we use. In his examination of combat data for the U.S. Civil War, WEISS [25] found that $\lambda \geq 100.0$ for both sides for all types of battles, but HELMBOLD [10, p. 39] has found values of $\lambda < 1.0$ for other sets of historical combat data.

In summary, in this section we have examined WEISS's [25] battle-termination model. We have seen that this particular model yields casualty-fraction breakpoints that are independent and exponentially distributed. We finally observe that a uniformly-distributed breakpoint corresponds to the case of "greatest uncertainty" in a side's engagement-termination process, i.e. the "most random" state of nature.

3.12. WEISS's Model of Engagement Outcomes in the U.S. Civil War.

As we have stressed above (see, for example, Section 2.8), the determination of battle outcome depends not only on the dynamics of combat (i.e. the model of the force-attrition process) but also on the battle-termination process. Thus, in order to obtain a complete model of engagement outcomes, one must add a model of force attrition to WEISS's above battle-termination model. This program was in fact carried out by H. K. WEISS in his very interesting and significant paper [25]. From examining battle-casualty data for the U.S. Civil War, WEISS [25] found that a FT|FT attrition process was suggested for combat attrition. The data also suggested a variety of exponential breakpoint for each side. WEISS then explored the consequences of these assumptions and found that the available historical combat data for the U.S. Civil War was in fairly good agreement with these hypotheses. Let us now examine his model in detail.

In addition to assumptions (A1) through (A3) given in the above section, WEISS assumed that combat attrition was a FT|FT process in "meeting engagements" (i.e. battles other than assaults on fortified lines). For the reader's convenience we collect here all the assumptions for WEISS's [25] model of engagement outcomes in the U.S. Civil War:

(A1) a side in combat considers only its own observed cumulative fractional loss (i.e. cumulative casualty fraction) to the moment of evaluation as the sole criterion for deciding whether or not to continue the battle; when the side has decided not to continue the engagement, it will abandon its mission and will try to break off the engagement,

(A2) battle termination in the future is independent of what has happened in the past (i.e. independence of nonoverlapping casualty-fraction intervals),

(A3) $P\left[\begin{array}{l|l} \text{a side terminates the battle} & \text{the side has continued to} \\ \text{at casualty fraction between} & \text{fight until casualty fraction} \\ f \text{ and } (f+\Delta f) & f \end{array}\right]$

$$= \lambda(f)\Delta f + O((\Delta f)^2) \ ,$$

(A4) for a given battle, the casualty-exchange ratio is constant, e.g. x_c/y_c = constant where (for example) x_c denotes X's cumulative casualties.[22]

Now, assumption (A4) implies that for a given battle

$$f_c^X = \gamma f_c^Y \ , \qquad (3.12.1)$$

where, for example, f_c^X denotes X's casualty fraction and

$$\gamma = \frac{a}{b} \cdot \left(\frac{y_0}{x_0}\right) . \qquad (3.12.2)$$

For a given battle, γ is constant, but it may vary in a random fashion from battle to battle. In other words γ is the realization of the random variable Γ and is realized before each battle. Accordingly, the modified ψ-function used for developing battle-outcome-prediction conditions is given by

$$\psi(t) = \begin{cases} \gamma t & \text{for } 0 \le t \le 1/\gamma, \\ 1 & \text{for } 1/\gamma \le t \ . \end{cases} \qquad (3.12.3)$$

Thus, we can invoke all the general results for the FT|FT attrition process with random breakpoints (see, for example, equations (3.8.6) through (3.8.10)).

As we have seen above in Section 3.11, assumptions (A1) through (A3) imply that each side's casualty-fraction breakpoint is independent of the other side's and has some type of exponential-like distribution (possibly with a variable termination rate). In other words, the casualty-fraction-breakpoint distribution, for example, for X must be of the form (see Section 3.11)

$$P[(F_c^X)_{BP} \le s]$$

$$= F_X(s) = \begin{cases} 1 - \exp\{-\int_0^s \lambda_X(\sigma)d\sigma\} & \text{for } 0 \le s < 1 , \\ 1 & \text{for } s = 1 . \end{cases} \tag{3.12.4}$$

Alternatively, we could have chosen

$$F_X(s) = \frac{1 - \exp\{-\int_0^s \lambda_X(\sigma)d\sigma\}}{1 - \exp\{-\int_0^1 \lambda_X(\sigma)d\sigma\}} . \tag{3.12.5}$$

We have already developed exact and approximate results for breakpoint distributions of the form (3.12.5) with $\lambda_X(f_c^X)$ = constant = λ_X. However, we will now develop results for breakpoint distributions of the form (3.12.4) (see also HELMBOLD [10]). In this case,

$$\bar{F}_X(s) = \begin{cases} \exp\{-\int_0^s \lambda_X(\sigma)d\sigma\} & \text{for} \quad 0 \leq s < 1 , \\ 0 & \text{for} \quad s = 1 . \end{cases} \tag{3.12.6}$$

WEISS [25] found that the Civil War combat data is fit by the following functional form of (3.12.6)

$$\ln\{1/\bar{F}_X(s)\} = \lambda_X s^3 ,$$

or

$$\bar{F}_X(s) = \begin{cases} \exp(-\lambda_X s^3) & \text{for} \quad 0 \leq s < 1 , \\ 0 & \text{for} \quad s = 1 . \end{cases} \tag{3.12.7}$$

Similarly,

$$\bar{F}_Y(t) = \begin{cases} \exp(-\lambda_Y t^3) & \text{for} \quad 0 \leq t < 1 , \\ 0 & \text{for} \quad t = 1 . \end{cases} \tag{3.12.8}$$

Although

$$\bar{F}_X(s) = [\exp(-\lambda_Y t^3)]^{\gamma^3 \lambda_X/\lambda_Y} = [\bar{F}_Y(t)]^a , \tag{3.12.9}$$

with

$$a = \gamma^3 \lambda_X/\lambda_Y , \tag{3.12.10}$$

the battle-outcome-prediction results of Table 3.II do not hold exactly, since the breakpoint distribution functions are not continuous.

Using (3.12.7) and (3.12.8) to evaluate the STIELTJES integral[23] (3.8.6), we obtain the following exact result for the probability that Y will win[24]

$$P_Y = \begin{cases} \dfrac{\lambda_X/\lambda_Y}{\lambda_X/\lambda_Y + 1/\gamma^3} \quad 1 - e^{-(\gamma^3\lambda_X+\lambda_Y)} & \text{for } 0 \le \gamma < 1 \,, \\ \dfrac{\lambda_X/\lambda_Y}{\lambda_X/\lambda_Y + 1/\gamma^3} \quad 1 - e^{-(\lambda_X+\lambda_Y/\gamma^3)} & \\ \qquad + e^{-(\lambda_X+\lambda_Y/\gamma^3)} & \text{for } 1 < \gamma \,, \end{cases} \tag{3.12.11}$$

so that we see that a good approximation to this exact result when λ_X, $\lambda_Y \geq 5.0$ is (see also Section 3.8).

$$\hat{P}_Y = \frac{(\lambda_X/\lambda_Y)}{[(\lambda_X/\lambda_Y) + 1/\gamma^3]} \,, \tag{3.12.12}$$

where $\hat{P}_Y$ denotes an approximate probability that Y wins. Let us also compute the average casualty fractions from (3.6.10), namely

$$\bar{f}_c^X = \gamma \bar{f}_c^Y = \int_0^u e^{-(\lambda_X+\lambda_Y/\gamma^3)s^3} \, ds \,, \tag{3.12.13}$$

where $u = \text{Minimum}(\gamma,1)$. The substitution $t = \lambda_Y(\lambda_X/\lambda_Y + 1/\gamma^3)s^3$ transforms (3.12.13) into

$$\bar{f}_c^X = \lambda_Y^{-1/3}(\lambda_X/\lambda_Y + 1/\gamma^3)^{-1/3} \int_0^q t^{-2/3} \, e^{-t} \, dt \,, \tag{3.12.14}$$

where $q = \lambda_Y(\lambda_X/\lambda_Y + 1/\gamma^3)$ Minimum$(\gamma^3,1)$, whence

$$\bar{f}_c^X = \lambda_Y^{-1/3}(\lambda_X/\lambda_Y + 1/\gamma^3)^{-1/3}\,\Gamma(\tfrac{4}{3}) - S\,, \qquad (3.12.15)$$

where

$$S = \frac{1}{3}\lambda_Y^{-1/3}(\lambda_X/\lambda_Y + 1/\gamma^3)^{-1/3}\int_q^\infty t^{-2/3}\,e^{-t}\,dt\,. \qquad (3.12.16)$$

Observing that $q \geq 1$ for $\lambda_X, \lambda_Y \geq 1$, we find that S has the following bound for $\lambda_X, \lambda_Y \geq 1$

$$S < \frac{1}{3}(\lambda_X + \lambda_Y/\gamma^3)^{-1/3}\,e^{-q} < \frac{1}{3}\,e^{-\lambda}\,, \qquad (3.12.17)$$

where $\lambda = \text{Minimum}(\lambda_X,\lambda_Y)$. Thus, we have WEISS's [25] approximation for $\lambda_X, \lambda_Y \geq 5.0$ (here $S < 0.0013$)

$$\gamma^{-1/2}\,\bar{f}_c^X = \gamma^{1/2}\,\bar{f}_c^Y = \frac{\Gamma(4/3)}{(\lambda_X\gamma^{3/2} + \lambda_Y\gamma^{-3/2})^{1/3}}. \qquad (3.12.18)$$

Thus, relatively simple approximate results are available for WEISS's model of engagement outcomes in the U.S. Civil War when $\lambda_X, \lambda_Y \geq 5.0$.

Thus we have completed our examination of the theoretical basis of WEISS's [25] model of engagement outcomes in the U.S. Civil War. The model is based on assumptions (A1) through (A4) above, which delineate the battle-termination process and the combat dynamics. We have subsequently deduced both exact and approximate results for the following quantities:

(Q1) probability of winning,

and (Q2) the average casualty fractions.

In the next section, we will see how WEISS's model provides a theoretical framework for analyzing combat data for the U.S. Civil War. Furthermore, agreement between historical combat data and theoretical predictions by the model is reasonably good.

3.13. WEISS's Empirical Examination of Engagement Outcomes in the U. S. Civil War.

In a very significant (and now classic) paper [25], H. K. WEISS has examined combat data on the U. S. Civil War to determine the extent to which it can be explained by simple mathematical relations, and he found some support (as well as problems) for such modelling. As we have seen above in Section 2.7, such attempts at empirical verification of LANCHESTER-type combat models have been rare. Such work is very important, however, since it may establish a scientific basis for combat modelling.

WEISS's Civil War paper should be considered to be the culmination of his work on LANCHESTER-type models of combat. This paper more fully develops ideas expressed in some of his earlier work [22-24]; namely, empirical investigation of the applicability of LANCHESTER-type equations to real combat [22-23] and the modelling of battle termination as a MARKOV process [24]. WEISS [23, p. 84] had earlier pointed out that an important question for military OR is whether, on the average, the outcome of actual combat tends to follow the linear law (2.4.3) or the square law (2.2.5). Additionally, some type of engagement-termination model is necessary for military analysis of combat (as we have repeatedly noted above). In [25] WEISS considered battle termination to be a MARKOV process: each time that a side sustains a casualty, its commander decides whether or not to continue the battle. He had previously used this type of model for the termination of a war. Additionally, WEISS's investigations [24 - 25] are more or less the point of departure for HELMBOLD's work.

WEISS's paper [25] began with a brief review of the overall characteristics of the U. S. Civil War with respect to the sizes of the forces involved, number of battles, total losses, and some distributional data on force ratios and casualty ratios. It then examined the applicability of

LANCHESTER's classic "combat laws." WEISS showed that a categorization of battles into "assaults on fortified lines" and "other battles" is significant. He then developed a mathematical relation between a side's fractional loss and its ability to continue a battle, and he showed that this describes fairly well the probability of winning a battle as a function of the relative casualty rates. The influence of fortification was demonstrated (although we will not examine it here). Finally, WEISS suggested some areas for future analysis.

After a discussion of the sources, availability, and quality of combat data for the U. S. Civil War, WEISS [25, pp. 765-766] examined data on force ratios in battles. Next, he examined the influence of the force ratio on the exchange ratio in order to establish the nature of the combat dynamics. WEISS sought to establish whether or not a simple form of LANCHESTER-type equations (i.e. LANCHESTER's classic combat formulations (2.2.1) and (2.4.1)) is consistent with the available historical combat data. Let us recall (<u>see</u>, for example, Table 2.XX) that <u>LANCHESTER</u>'s <u>Square Law</u> $b(x_0^2 - x^2) = a(y_0^2 - y^2)$ implies that the <u>overall casualty-exchange ratio should be inversely proportional to the "average" force ratio</u>, namely

$$\frac{x_c}{y_c} = \frac{a}{b(\bar{x}/\bar{y})} , \tag{3.13.1}$$

where $x_c = x_0 - x_f$ denotes X's casualties in the battle, $\bar{x} = (x_0+x_f)/2$ denotes X's average force level, b represents X's fire effectiveness, and similarly for the Y quantities y_c, $\bar{y}$, and a. If combat were to obey LANCHESTER's Square Law, then we would expect the loss ratio to be

strongly correlated with the force ratio (see WEISS [23, pp. 84-87]). On the other hand, LANCHESTER's Linear Law $b(x_0-x) = a(y_0-y)$ implies that the overall casualty-exchange ratio should be independent of the (initial) force ratio, namely

$$\frac{x_c}{y_c} = \frac{a}{b} . \qquad (3.13.2)$$

From an examination of a scatter diagram (with coordinates of casualty ratio and loss ratio) for "all" battles, WEISS [25, p. 768] found no apparent correlation of the casualty ratio with the force ratio. He then aggregated battles into two classes: (I) "attacks on fortified lines," and (II) all other battles, called for convenience "meeting engagements."

Figure 3.18 shows the loss ratio plotted against the initial force ratio for "attacks on fortified lines." The data shows considerable scatter with no pattern immediately discernable. Examination of battles other than assaults on fortified lines (denoted by WEISS as "meeting engagements") yield a "more homogeneous picture" [25, p. 770], as shown in Figure 3.19. In this figure there is no obvious correlation between the casualty ratio and the initial force ratio, suggesting that LANCHESTER's Linear Law might be justified here. WEISS noted that the "scatter in exchange ratio from battle to battle amounts to only a factor of two at most." Hence, one is led to postulate that attrition in Civil War "meeting engagements" followed FT|FT attrition, with the casualty-exchange ratio $x_c/y_c = a/b$ being a random variable realized before each individual battle.

WEISS [25, pp. 770-775] went on to examine data for "meeting engagements" in greater detail. He found for a somewhat limited sample

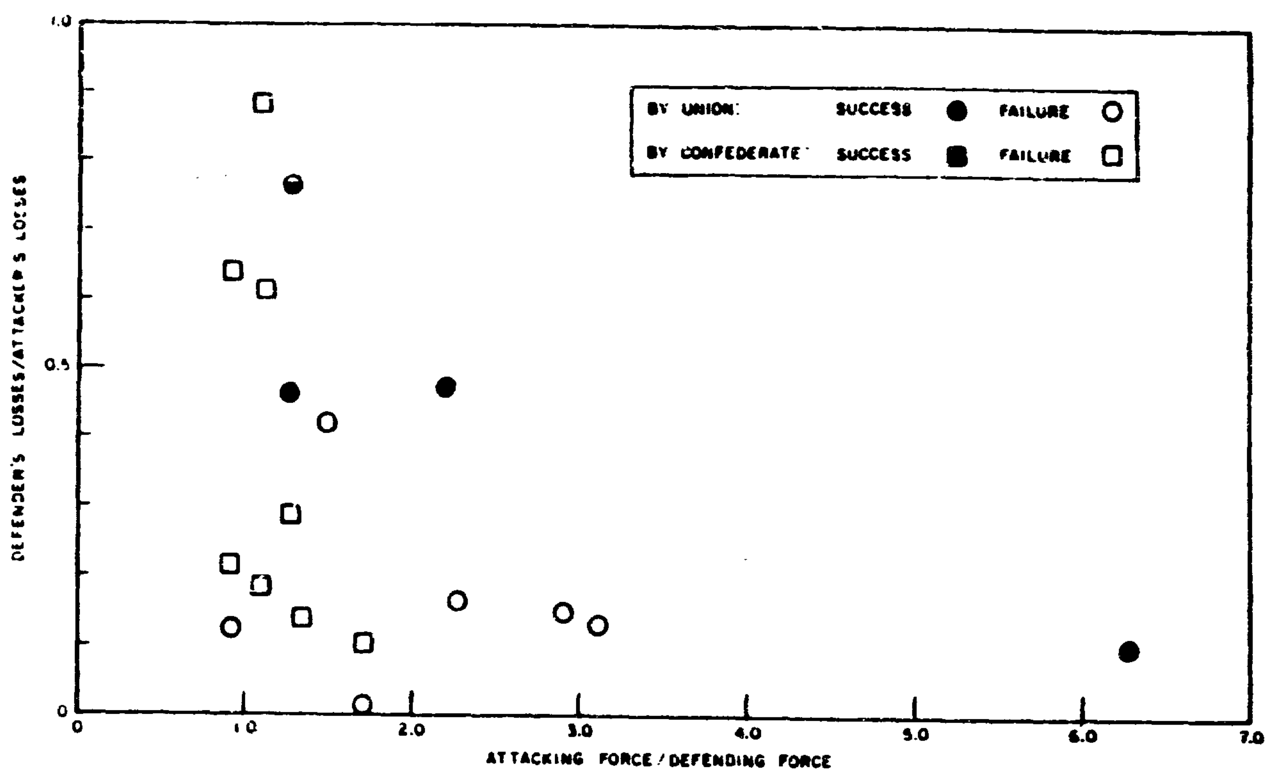


Figure 3.18. Initial force ratios and loss ratios in "attacks on fortified lines" for U. S. Civil War (from WEISS [25]).

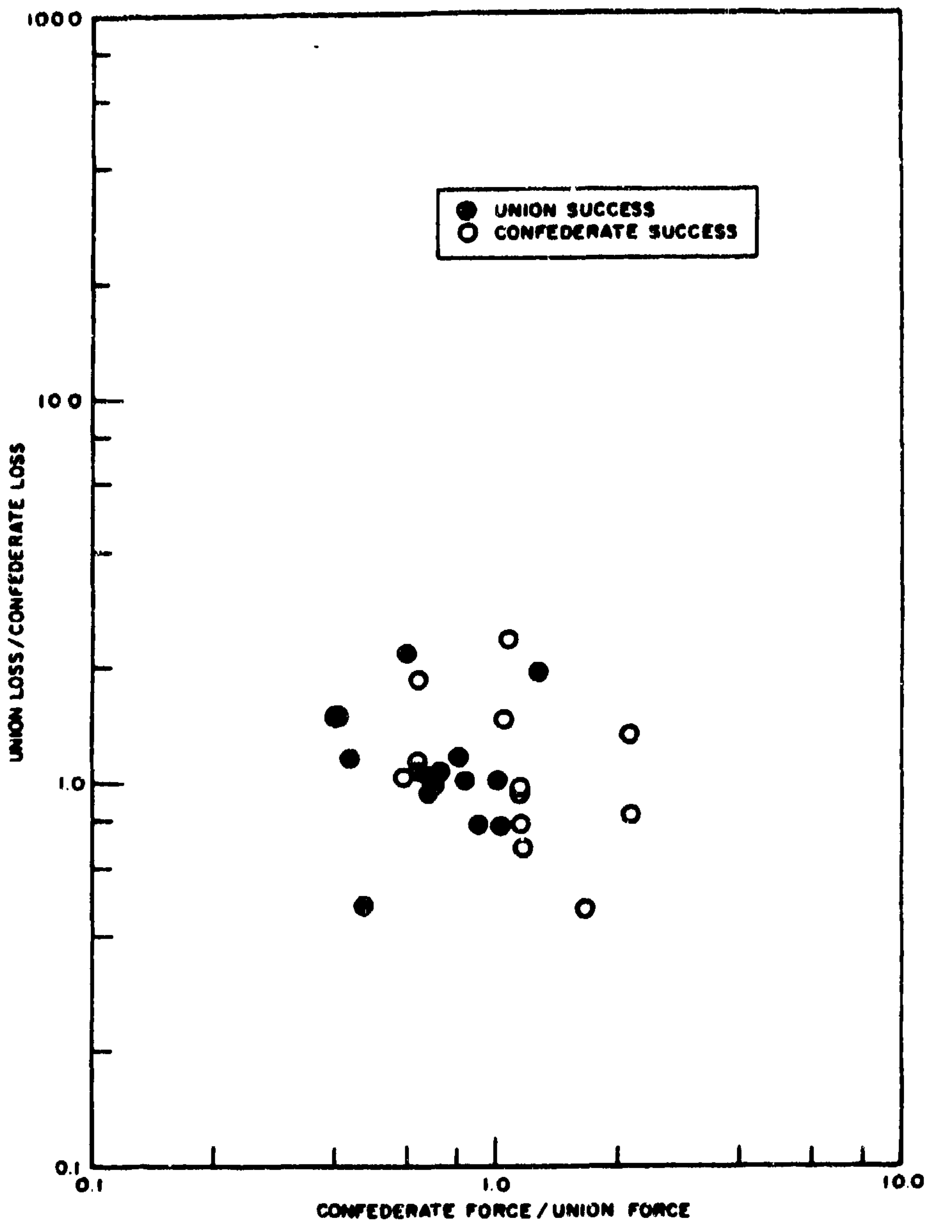


Figure 3.19. Initial force ratios and loss ratios in battles other than assaults on fortified lines for U. S. Civil War (from WEISS [25]).

(28 battles in all) that the probability of winning was strongly related to the initial force ratio, as shown in Table 3.VII and Figure 3.20. Also shown as the smooth curve in Figure 3.20 is the function

$$P = 1/(1 + \mu^3) , \tag{3.13.3}$$

where P denotes the probability of a Union win and μ = initial force ratio: Confederate/Union. WEISS also found that casualties tended to be about equal and the larger force tended to win. This suggested to him a combat model based on ability to continue fighting as a function of sustained fractional losses. The end result was WEISS's model for combat outcomes, which we have considered in the previous section. His model was based on the following empirical findings for "meeting engagements" in the U. S. Civil War:

(F1) the probability of winning appeared to be a strong function of the initial force ratio,

(F2) casualty ratios were independent of which side attacked, or who won, or the initial force ratio, and they lay within the extreme values 0.46 and 2.33,

(F3) on the average the loser sustained 15 per cent casualties; the winner, 12 per cent.

The above is the motivation for WEISS's model of engagement outcomes, which we have studied in Section 3.12. The main information extracted by WEISS from his model was (1) the probabilities of winning, and (2) the average casualty fractions. WEISS estimated model parameters (the fractional loss ratio, γ, and the battle-termination rates, λ_X and λ_Y) from the combat data in the following fashions. He first estimated,

TABLE 3.VII. Fraction of Union (X) Wins in "Meeting Engagements" as a Function of the Initial Force Ratio for U.S. Civil War (from WEISS [25]).

Initial Force Ratio: Confederate/Union Strength	Number of Cases	Fraction of Union Wins		
		Average	50% Confidence Limits	
0.40 - 0.49	3	1.00	1.00	0.63
0.50 - 0.79	11	0.68	0.80	0.54
0.80 - 1.25	11	0.50	0.64	0.36
1.26 - 2.00	1	0	0.75	0
2.01 - 2.50	2	0	0.50	0
	28			

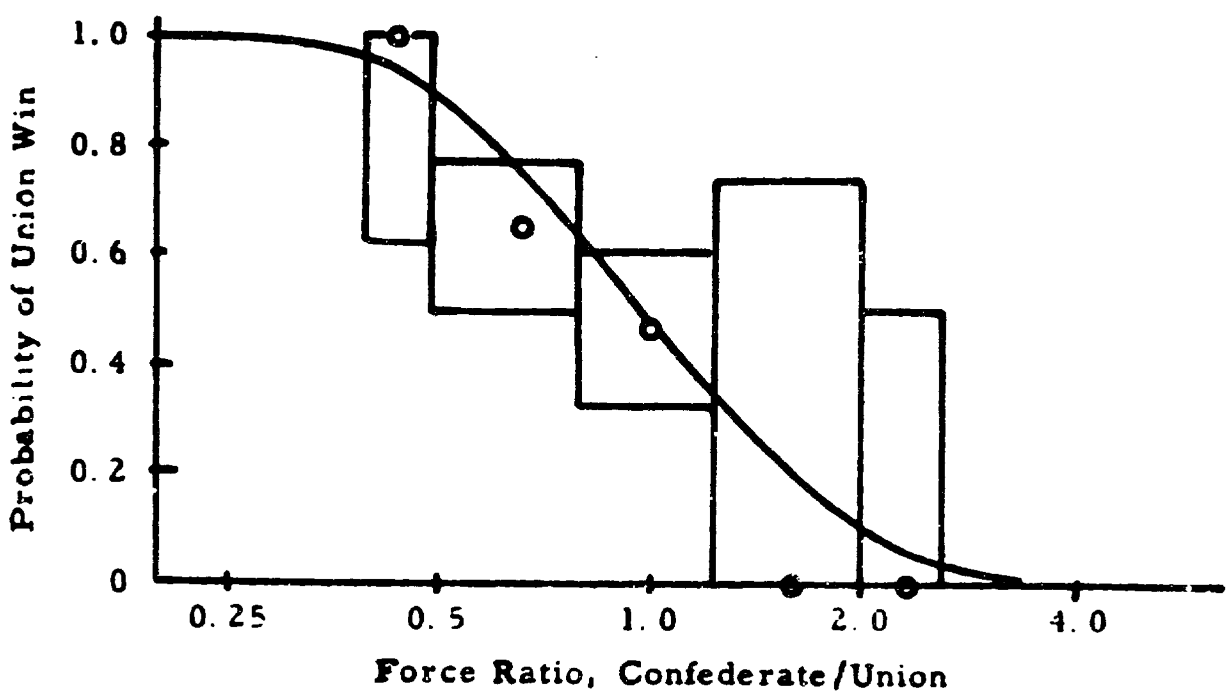


Figure 3.20. Probability of a Union win versus the initial force ratio for "meeting engagements" (28 battles) in the U. S. Civil War (from WEISS [25]).

for example, $\Phi_X(f_c^X) = \bar{F}_X(f_c^X) = P$[X fights at least until casualty fraction > f_c^X] by doing the following:

(T1) rank all battles in order of increasing $(f_c^X)_f$, where $(f_c^X)_f$ denotes the fractional loss of X at the time the battle ended, regardless of who won the battle,

(T2) estimate probability of continuing battle, $\Phi_X(f_c^X)$, from the formula

$$\hat{\Phi}_X = \hat{\Phi}[(f_c^X)_f] = O_X/(N_0-L_Y) , \qquad (3.13.4)$$

where $O_X = O_X[(f_c^X)_f]$ denotes the observed number of battles that lasted until $(f_c^X)_f$, N_0 denotes the total number of battles, and L_Y denotes the number of battles lost by Y at lesser value of $(f_c^X)_f$.

The estimate (3.13.4) for $\Phi_X(f_c^X)$ may be justified as follows. If we forget about Y losing (i.e. Y deciding to quit), then we would expect (on the average) $N_0 \cdot \Phi_X(f_c^X)$ battles to continue past f_c^X. However, L_Y battles ended at lesser value of f_c^X, since Y decided to terminate the engagement. If Y had not decided to terminate these engagements, then $L_Y \cdot \varphi_X(f_c^X)$ battles would have continued at least until f_c^X. In other words,

$$O_X = N_0 \cdot \hat{\Phi}_X(f_c^X) - L_Y \cdot \hat{\Phi}_X(f_c^X) ,$$

whence follows (3.13.4). We can similarly estimate $\Phi_Y(f_c^Y) = \bar{F}_Y(f_c^Y)$ from the formula

$$\hat{\Phi}_Y = \hat{\Phi}_Y[(f_c^Y)_f] = O_Y/(N_0 - L_X) \quad . \tag{3.13.5}$$

In other words, from tabulations of the final casualty fractions $(f_c^X)_f$ and $(f_c^Y)_f$ in each battle, one can estimate values for the casualty-fraction-breakpoint complementary distribution functions $\Phi_X(f_c^X) = \overline{F}_X(f_c^X)$ and $\Phi_Y(f_c^Y) = \overline{F}_Y(f_c^Y)$.

WEISS [25, p. 778] plotted $\ell n(1/\Phi)$ against f_c and found that

$$\ell n(1/\Phi) = kf_c^3 \,, \qquad k = 150, \tag{3.13.6}$$

gave a good fit to the data for both sides (see Figure 3.21), although the fit could be improved by considering different functions for the two sides. In other words, the Civil War data suggested that for "meeting engagements"

$$\overline{F}_X(s) = \begin{cases} \exp(-ks^3) & \text{for} \quad 0 \le s < 1 \,, \\ 0 & \text{for} \quad s = 1 \,, \end{cases} \tag{3.13.7}$$

and

$$\overline{F}_Y(t) = \begin{cases} \exp(-kt^3) & \text{for} \quad 0 \le t < 1 \,, \\ 0 & \text{for} \quad t = 1 \,, \end{cases} \tag{3.13.8}$$

where $\overline{F}_X(s) = \Phi_X(f_c^X)$, etc. Consequently, for "meeting engagements" in the U. S. Civil War, battle outcome may be approximately modelled by (see Section 3.12)[25]

$$\hat{P}_X = 1/(1 + \gamma^3) \,, \tag{3.13.9}$$

and

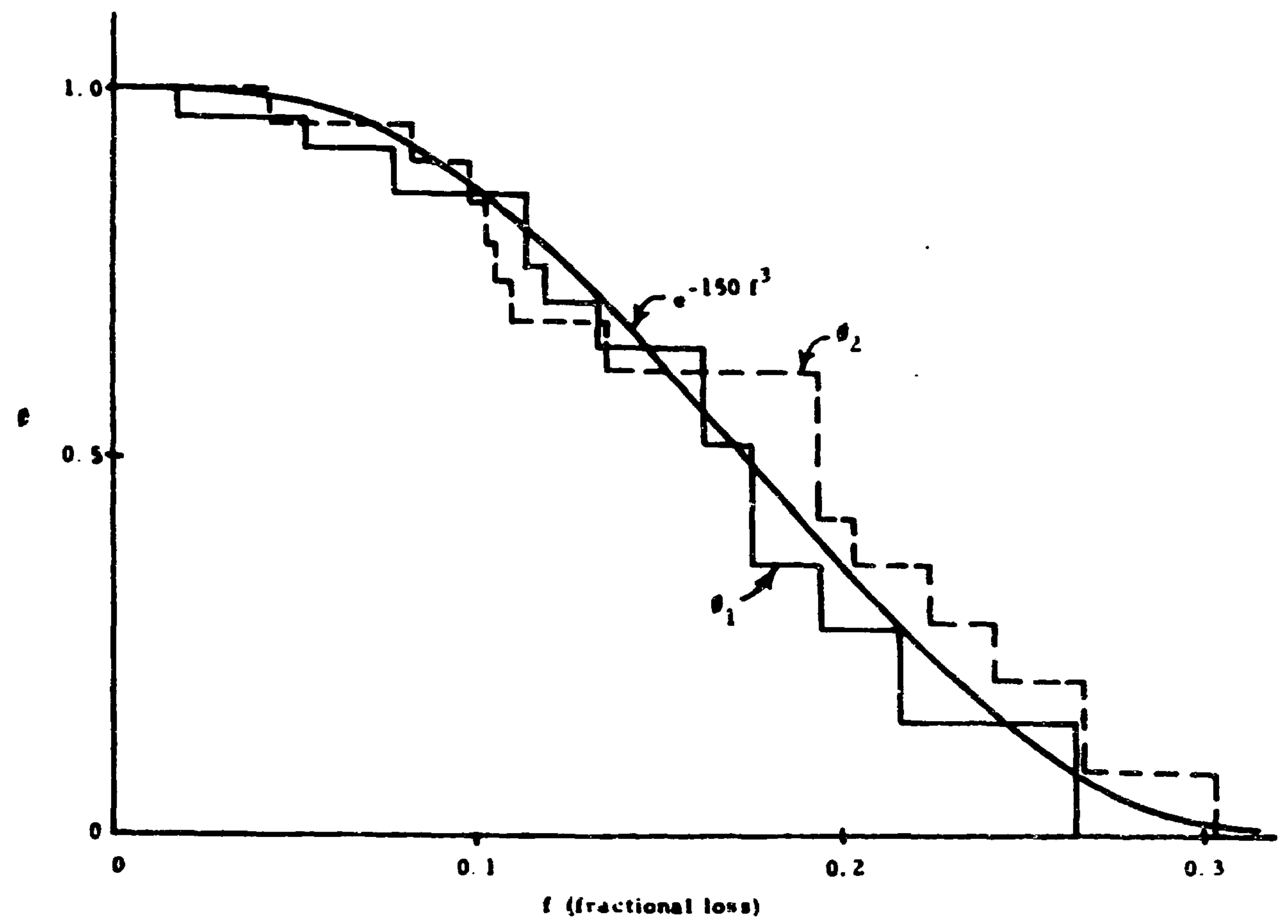


Figure 3.21. Probability of continuing battle versus loss fraction for "meeting engagements" in the U. S. Civil War (from WEISS [25]).

$$\gamma^{-1/2}\,\hat{f}_c^X = \gamma^{1/2}\hat{f}_c^Y = \Gamma(4/3)/(k\gamma^{3/2} + k\gamma^{-3/2})^{1/3}\ , \qquad (3.13.10)$$

where $\gamma = f_c^X/f_c^Y$ denotes the fractional-loss ratio (also referred to elsewhere as the "normalized" initial force ratio). WEISS [25, p. 780] also subdivided the battle data by ranges of γ, i.e. ranges of the fractional-loss ratio, into four equal groups and compared (3.13.9) and (3.13.10) with the averages of the data in each group. Results are shown in Table 3.VIII.

Thus, WEISS [25] developed a model of engagement outcomes in the U. S. Civil War, and this model yielded theoretical predictions that were in fairly good agreement with the historical data. Whether or not such a model is generally applicable to modern warfare was, however, not decided by WEISS's investigation (and, indeed, it cannot be so decided). Moreover, WEISS clearly states that the purpose of his investigation [25] was to indicate some of the factors possibly involved in modelling combat operations and to stimulate further research on the scientific study of warfare. WEISS's work [25] provides many of the ideas upon which HELMBOLD's investigation [10] of various breakpoint hypotheses is based.

One final point, however, merits further discussion, and that is the general nature of scientific verification of a combat model. The process is an indirect one in which we must deduce testable consequences from the modelling assumptions (or hypotheses). Furthermore, we can never "prove that a model is true," but we can sometimes determine that a model yields theoretical implications (i.e. consequences) that are at variance with available empirical evidence. In this case, we should reject the model as being untenable in light of empirical evidence and seek alternative tenable hypotheses.

TABLE 3.VIII. Comparison of Historical Combat Data with Model Results for "Meeting Engagements" in the U.S. Civil War (from WEISS [25]).

Range of γ	0.22-0.69	0.70-0.79	0.80-1.11	1.12-2.78
Average γ	0.56	0.73	0.94	1.91
Number of battles	7	7	7	7
Fraction of Union (X) wins	0.79	0.57	0.50	0.29
$\hat{P}_X$ from ave. γ [eq. (3.13.9)]	0.85	0.72	0.55	0.13
Average Confederate (Y) fractional loss	0.17	0.19	0.11	0.08
$\hat{\bar{f}}_c^Y$ from eq. (3.13.10)	0.16	0.15	0.13	0.08

NOTE: Model parameters, i.e. $\hat{\lambda}_X = \hat{\lambda}_Y = k$, are estimated from same data for which the above comparison is made.

3.14. HELMBOLD's Empirical Investigation of the Validity of Breakpoint Hypotheses.

Finally, we come to HELMBOLD's work [10] concerning the scientific validity of a certain breakpoint hypothesis and several variants thereof. This work should be considered to be the extension and synthesis of earlier work by both H. K. WEISS [22-25] and HELMBOLD himself [8-9]. HELMBOLD's RAND report [10] is in our opinion probably the most significant piece of work on either the scientific aspects of modelling engagement termination or the scientific evaluation of the validity of combat models.

This remarkable report first establishes a rather comprehensive theoretical framework for modelling engagement termination and then deduces testable consequences of the modelling hypotheses (i.e. the assumptions). These consequences are then compared with empirical evidence (i.e. historical combat data) to evaluate the model's scientific validity. HELMBOLD found that his basic breakpoint hypothesis was at variance with empirical evidence, but he advanced no alternate hypothesis that he felt was satisfactory. Consequently, this fine work has apparently had little influence on the combat-modelling community.

HELMBOLD's work on modelling battle termination [10] has greatly influenced this chapter. His theoretical treatment of modelling battle termination is both significant and interesting. Moreover, this work contains the germs of many ideas that are significant in their own right (e.g. the approach used in Section 3.5 for developing victory-prediction conditions[26]).

HELMBOLD's basic breakpoint hypothesis [10] consisted of the following three assumptions:

(A1) each side independently selects a breakpoint from a distribution of such breakpoints and gives up the battle when its casualty fraction reaches this breakpoint,

(A2) these breakpoint distribution curves are generally applicable,

(A3) the casualty fractions of the forces are deterministically and monotonically related to each other via the ψ-function, i.e. $f_c^X(t) = \psi[f_c^Y(t))]$ for $0 \leq t \leq t_f$.

How does one go about showing whether or not HELMBOLD's breakpoint hypothesis is true[27]? This task indeed appears formidable, since the assumptions (A1) through (A3) do not specify either (1) a specific breakpoint distribution for each side, or (2) a specific ψ-function, which relates the two casualty fractions by $f_c^X = \psi(f_c^Y)$. HELMBOLD has overcome this difficulty brilliantly: he has shown how observed casualty-fraction distributions can be used to test the breakpoint hypothesis. More precisely, HELMBOLD [10, pp. 16-17] has shown how the ψ-function and its inverse ψ^{-1} may be determined from the casualty-fraction conditional distributions. In other words, one may estimate ψ and ψ^{-1} (denote these estimated functions as $\hat{\psi}$ and $\hat{\psi}^{-1}$, respectively) from available historical combat data by developing casualty-fraction conditional distributions.

Having determined $\hat{\psi}$ and $\hat{\psi}^{-1}$ by HELMBOLD's graphical procedure, we may plot the estimated functions on a graph and see whether or not they are indeed inverse functions, i.e. whether or not $\hat{\psi}$ is a reflection of $\hat{\psi}^{-1}$ in the 45 degree line through the origin (see Figure 3.22). If $\hat{\psi}$ and $\hat{\psi}^{-1}$ satisfy the inverse functional relationship, then this evidence would lend support to HELMBOLD's breakpoint hypothesis. If $\hat{\psi}$ and $\hat{\psi}^{-1}$ do not satisfy the necessary mathematical relationship between inverse functions, then HELMBOLD's breakpoint hypothesis would be definitely disproven. In other words, that $\hat{\psi}$ and $\hat{\psi}^{-1}$ as developed from the casualty-fraction conditional distributions should be inverse functions is a testable

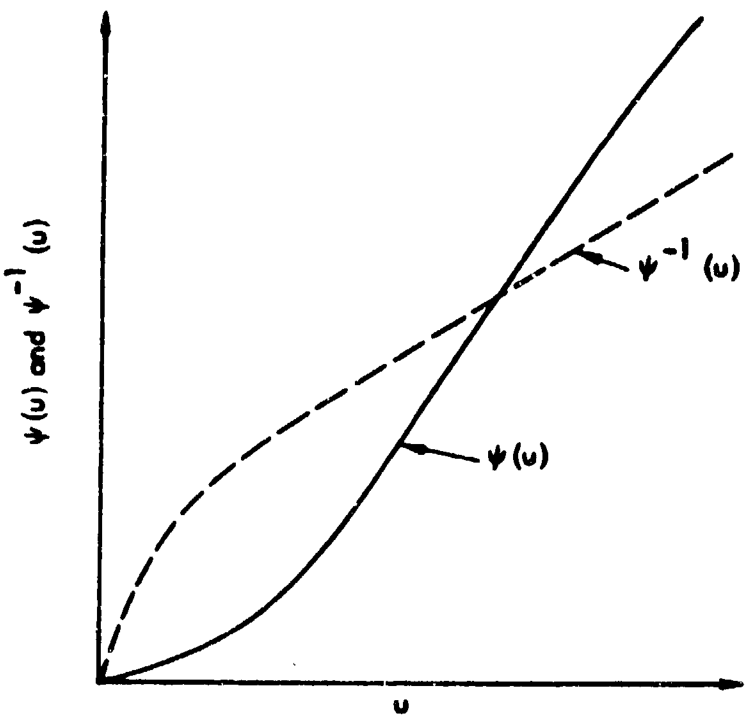


Figure 3.22. Inverse functional relationship that must hold between $\psi(u)$ and $\psi^{-1}(u)$ (from HELMBOLD [10]).

consequence of HELMBOLD's breakpoint hypothesis (i.e. assumptions (A1) through (A3) above). The validity of the breakpoint hypothesis may therefore be tested by seeing whether or not actual combat data has this inverse-function property inherent in it.

Let us now show how ψ and ψ^{-1} may be determined from the casualty-fraction conditional distributions. For convenience let us introduce the following notation for the casualty-fraction conditional distributions

$$\Delta_{XX}(s) = P[(F_c^X)_f \leq s | X \text{ wins}] \ , \tag{3.14.1}$$

and

$$\Delta_{YX}(t) = P[(F_c^Y)_f \leq t | X \text{ wins}] \ . \tag{3.14.2}$$

It follows that

$$\begin{aligned} \Delta_{XX}(s) &= P[(F_c^X)_f \leq s | X \text{ wins}] \\ &= P[(F_c^Y)_f \leq \psi^{-1}(s) | X \text{ wins}] = \Delta_{YX}(\psi^{-1}(s)) \ , \end{aligned} \tag{3.14.3}$$

since $F_c^X = \psi(F_c^Y)$. In other words, Δ_{XX} and Δ_{YX} have the same value for s and $\psi^{-1}(s)$, respectively. We may similarly define $\Delta_{YY}(t)$ and $\Delta_{XY}(s)$, and it follows that

$$\Delta_{YY}(t) = \Delta_{XY}(\psi(t)) \ . \tag{3.14.4}$$

Now suppose that we had a graphical plot of the observed casualty fraction for a set of battles won by X (chosen to be the attacker). Such a hypothetical plot is shown in Figure 3.23. Using (3.14.3) and a plot like that shown in Figure 3.23, we may graphically determine the estimated value of $\psi^{-1}(s)$, denoted as $\psi^{-1}(s)$, by repeatedly determining $\hat{\psi}^{-1}(s_1)$ for

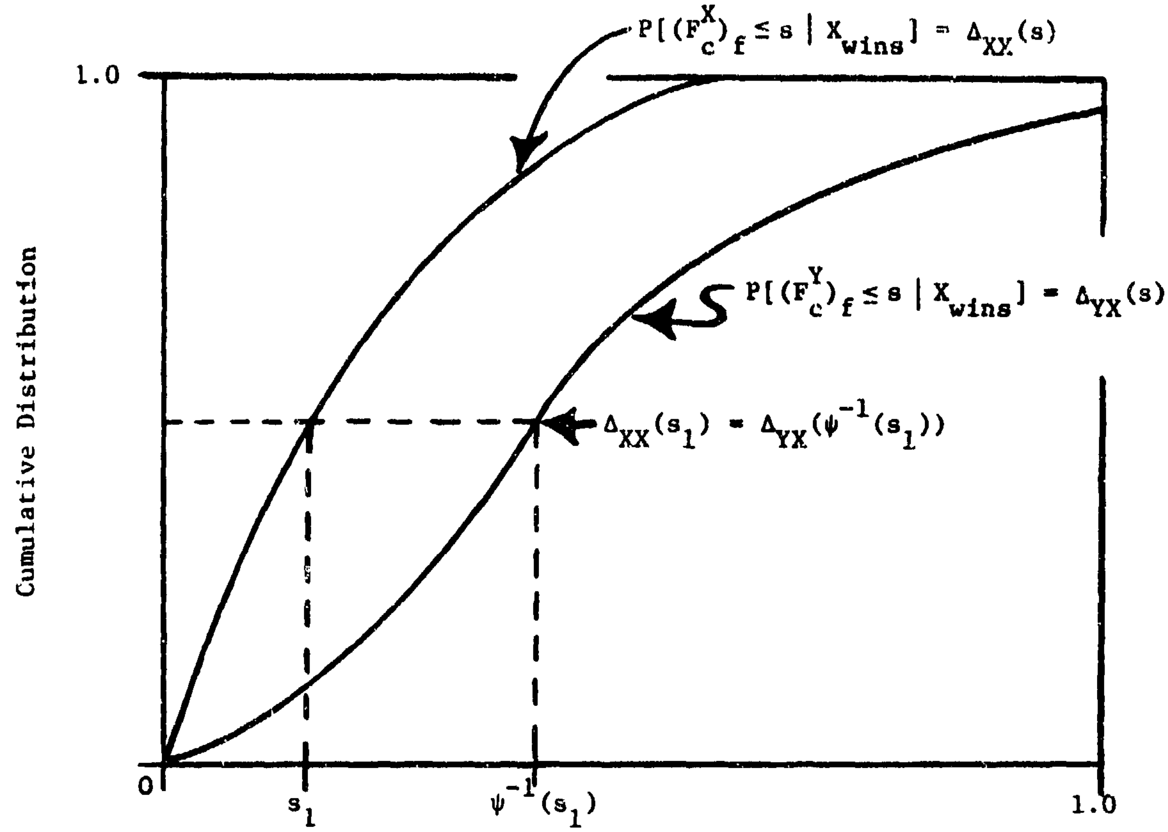


Figure 3.23. Use of hypothetical casualty-fraction distributions for battles won by X (the attacker) to determine $\hat{\psi}^{-1}(s)$.

several different given values of X's casualty fraction, s_1. In a similar fashion, we may graphically determine $\hat{\psi}(t)$ from the observed casualty-fraction distributions when Y wins.

HELMBOLD [10] also developed certain important relationships between the casualty-fraction conditional distributions. These relationships played a key role in the testing of his breakpoint hypotheses. They are contained in the following two propositions (for proofs, see HELMBOLD [9]).

> PROPOSITION 3.14.1: If $\Delta_{XX}(u) = \Delta_{YY}(u)$ and $\Delta_{YX}(u) = \Delta_{XY}(u)$, then $\psi = \psi^{-1} = I$, where I denotes the identity function.

> PROPOSITION 3.14.2: If $\psi(s) \geq s$ for some s, then $\Delta_{YY}(s) \geq \Delta_{XY}(s)$ and $\Delta_{XX}(s) \leq \Delta_{YX}(s)$. Conversely, if $\psi(s) \leq s$ for some s then $\Delta_{YY}(s) \leq \Delta_{XY}(s)$ and $\Delta_{XX}(s) \geq \Delta_{YX}(s)$.

The first result (i.e. Proposition 3.14.1) says that if the winner and the loser have the same casualty-fraction distributions (regardless of whether the attacker X or the defender Y wins the battle), then the ψ-function must be the identity function (i.e. fractional casualties are exchanged equally). The second result (i.e. Proposition 3.14.2) may be used to develop the possible types of relations between the casualty-fraction distributions (see HELMBOLD [10, pp. 33-34]).

HELMBOLD [10] went on to test in the manner outlined above his breakpoint hypothesis by comparing the model's consequences (namely, that the graphically determined functions $\hat{\psi}$ and $\hat{\psi}^{-1}$ must satisfy the necessary inverse functional relationships) with available combat data. HELMBOLD used

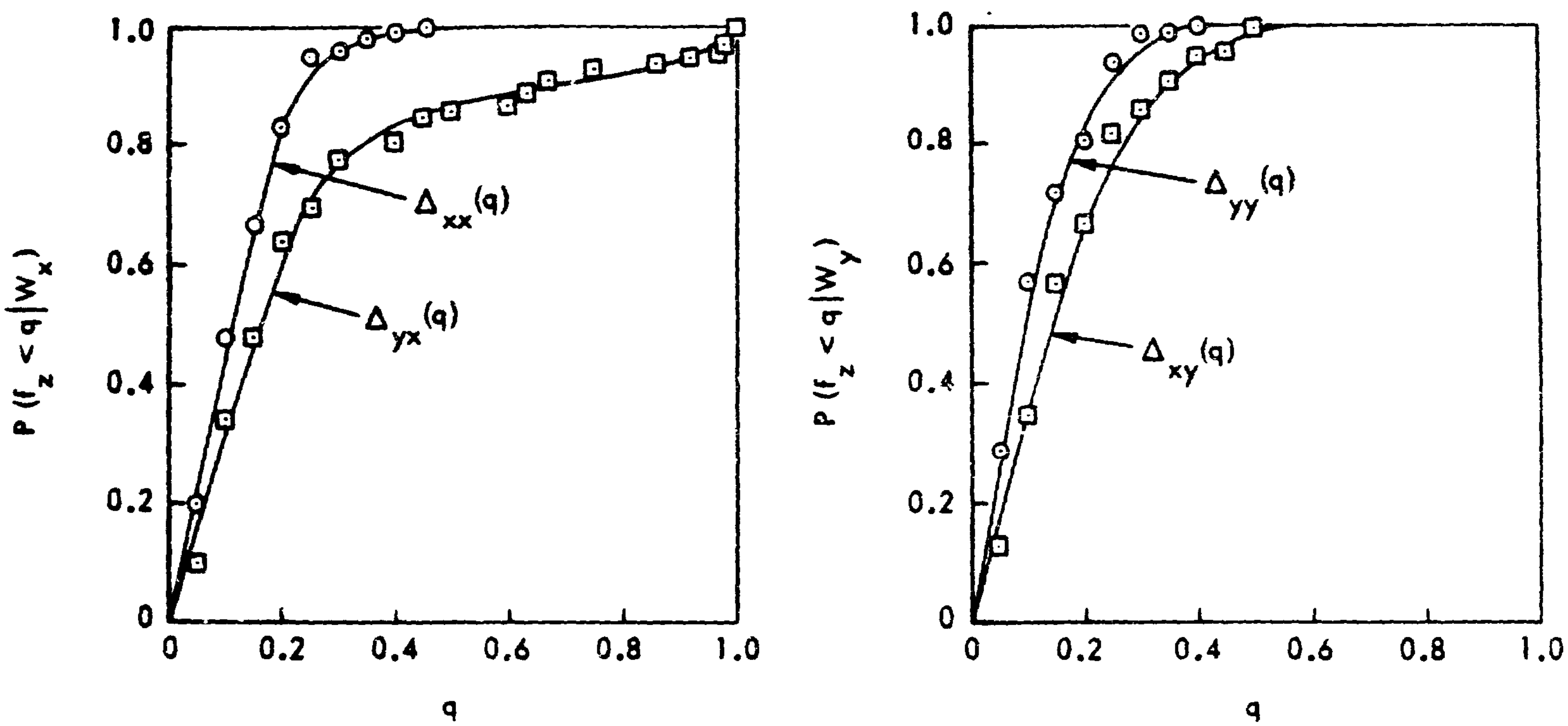
several sets of empirical data on casualty-fraction distributions in this work: namely,

(S1) data from his earlier empirical work (<u>see</u> HELMBOLD [8-9]).

(S2) data extracted from BODART's <u>Kriegs-Lexicon</u> [2] by WILLARD [26], as modified by SCHMIEMAN [16].

Let us first consider HELMBOLD's comparison of his model with the data base generated by his earlier work [8-9]. From the raw combat data (i.e. the initial and final force levels for the attacker and the defender), HELMBOLD obtained the casualty-fraction conditional distributions shown in Figure 3.24 (<u>see</u> HELMBOLD [10, pp. 21-22] for more detailed casualty-fraction data). The values for $\hat{\psi}(q)$ and $\hat{\psi}^{-1}(q)$ read graphically from the plots of this figure (according to the procedure described above) are plotted in Figure 3.25. From this latter figure we see that $\hat{\psi}$ and $\hat{\psi}^{-1}$ are clearly not inverse functions, which is a necessary consequence of HELMBOLD's breakpoint hypothesis. This fact is strong evidence against this breakpoint hypothesis being true (<u>see</u> discussion above).

HELMBOLD [10, pp. 25-32] then performed the same test with a much larger sample of combat data, data extracted from BODART's <u>Kreigs-Lexicon</u> [2] by WILLARD [26] (as modified by SCHMIEMAN [16]). He considered three different groupings of this data: (I) the entire set of 1080 battles, (II) Category I battles (i.e. "open" battles in the sense that both sides could, with about equal facility, disengage and conduct an orderly withdrawal), and (III) Category II battles (i.e. "closed" battles in the sense that one side was encircled or otherwise in a position from which an orderly withdrawal could not be readily made, and whose options for maneuver were markedly more restricted than those of his opponent).



(a) Attacker wins

(b) Defender wins

Figure 3.24. Empirical casualty-fraction conditional distributions, composite of data from HELMBOLD [8-9] (figure from HELMBOLD [10]).

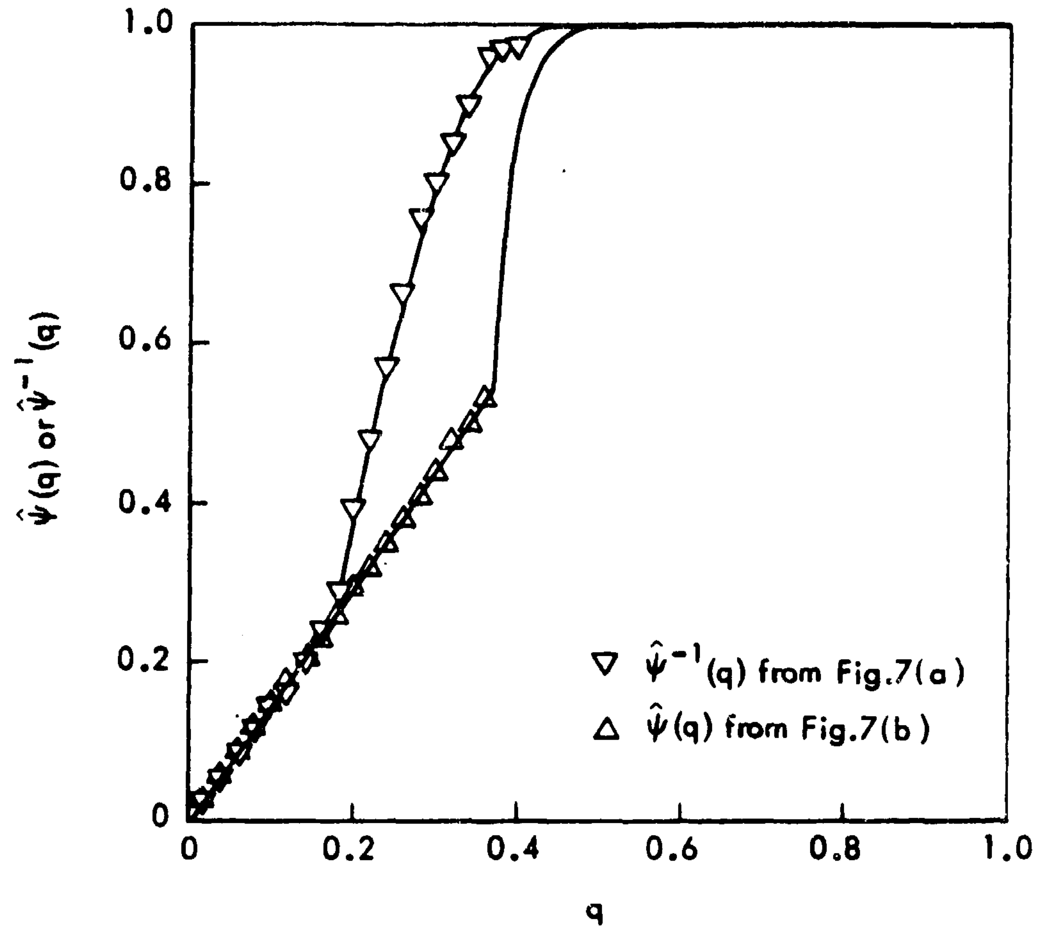


Figure 3.25. Values of $\hat{\psi}$ and $\hat{\psi}^{-1}$ derived from Figure 3.24 (from HELMBOLD [10]). HELMBOLD's Figure 7 is our Figure 3.24.

The overall BODART data showed [10, pp. 26-26] that the distribution of the attacker's (i.e. X's) casualties when the attacker won is about equal to the distribution of the defender's casualties when the defender won; and it showed that the distribution of the defender's casualties when the attacker won is about equal to the distribution of the attacker's casualties when the defender won, i.e.

$$\Delta_{XX}(u) = \Delta_{YY}(u) , \tag{3.14.5}$$

and

$$\Delta_{YX}(u) = \Delta_{XY}(u) . \tag{3.14.6}$$

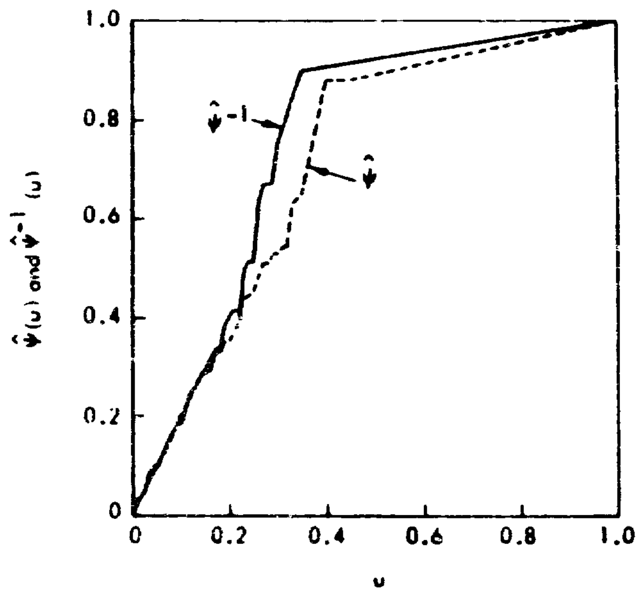
Proposition 3.14.1 says, however, that we should have

$$\psi = \psi^{-1} = I ,$$

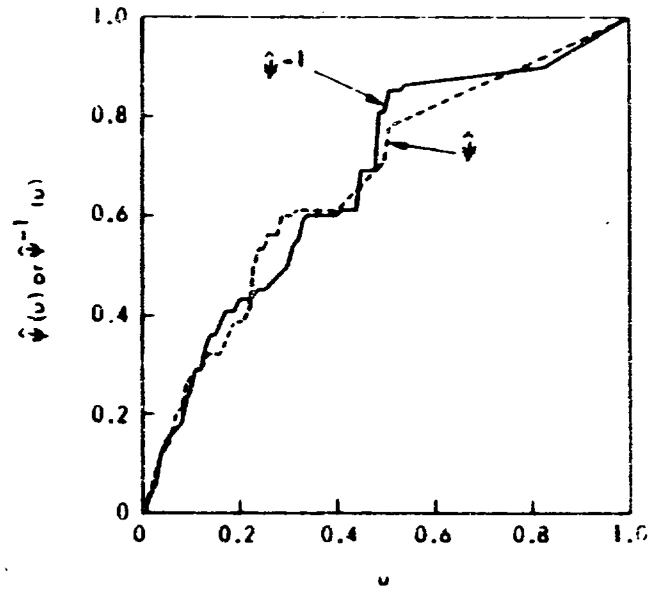
and this consequence was contradicted by the empirical evidence. HELMBOLD then examined data for only Category I battles and also data for only Category II battles and found further contradiction to the breakpoint hypothesis.

Thus, HELMBOLD [10] found that for all of the data sets that he analyzed, $\hat{\psi}$ and $\hat{\psi}^{-1}$ were clearly <u>not</u> mutually inverse mathematical functions as required by his original breakpoint hypothesis (<u>see</u> Figures 3.25 and 3.26). Consequently, this breakpoint hypothesis is not tenable. However, instead of being inverse functions it appears that

$$\hat{\psi} = \hat{\psi}^{-1} \neq I ,$$



(a) Category I battles



(b) Category II battles

Figure 3.26. Values of $\hat{\psi}$ and $\hat{\psi}^{-1}$ for Category I and Category II battle data (from HELMBOLD [10]).

but that (3.14.5) and (3.14.6) held, at least approximately, for all the data analyzed. HELMBOLD [10, p. 32] concluded that this latter empirical fact (i.e. (3.14.5) and (3.14.6) holding) was at the crux of the contradiction of the breakpoint hypothesis by the available historical combat data.

HELMBOLD [10, pp. 33-61] went on to consider some tentative modifications of his original breakpoint hypothesis and discussed them in terms of the light they shed on the prospects for developing a theory that would satisfactorily explain the available combat data. His modifications may roughly be stated as follows (<u>see</u> [10] for further details):

(Modification 1) Use one ψ-function when the attacker (X) wins, and a different ψ-function when the defender wins.

(Modification 2) Replace (A3) in HELMBOLD's original breakpoint hypothesis by the following: the casualty fractions of the forces engaged in a given battle are related to each other by

$$f_c^X = \gamma f_c^Y ,$$

where γ is the realization of the random variable Γ, which is realized before each battle, and Γ is log-normally distributed with mean zero and standard deviation of 0.76.

(Modification 3) Give up (A2) in HELMBOLD's original breakpoint hypothesis and let the break curves depend on the class or type of battle that is under study.

Each of these three tentative modifications was then shown by HELMBOLD to be unsatisfactory, i.e. yield some consequence contradicted by the available combat data. For example, Proposition 3.14.2 was used to show that the observed casualty-fraction distributions have shapes that are at variance with theoretical predictions. HELMBOLD finally outlined the properties that a satisfactory theory of battle termination should possess. All the above versions of the breakpoint hypothesis violates some of HELMBOLD's criteria.

Thus, HELMBOLD's principal finding [10, p. v] was that the breakpoint hypothesis yielded theoretical implications that are at variance with the available battle termination data in several essential respects. He also discussed the properties of a satisfactory theory of battle termination but did not develop such a theory that could satisfactorily account for the available data. HELMBOLD [10, p. v] felt that "until a better theoretical explanation of the battle termination process becomes available, the soundness of models of combat such as war games and computer simulations that make essential use of breakpoint hypotheses is suspect." Nevertheless, the assessment of the outcomes of tactical engagements does require the use of some type of engagement-termination conditions (i.e. battle-termination model). We (and the rest of the military OR community) will continue to use our Breakpoint Hypothesis (see Section 3.2 above) until a better alternative comes along.

FOOTNOTES for Chapter 3

1. For example, the eminent military historian T. N. DuPUY [11] (see also [6]) has considered the outcome of battle to be given by the formula

(outcome or result)

= (mission accomplishment) + (space effectiveness) + (casualty effectiveness).

Here we should take "space effectiveness" to mean "possession of the battlefield." There are, of course, difficulties in quantifying the above three concepts of mission accomplishment, etc. (one should "operationally" define them). Nevertheless, it is significant that historically it is unclear who was the "winner" for many battles. However, combat models, which are supposed to be representations of the real world, usually give quite clear-cut results. This shortcoming, moreover, is not just limited to LANCHESTER-type models of warfare.

2. In her classic and definitive study of the effects of casualties on combat effectiveness, D. CLARK [5] has considered casualty and replacement data from the so-called morning reports of 44 infantry battalions taking part in seven engagements in World War II in the European theater of operations. In this rare empirical study, she considered the following three categories of breakpoints:

I. attack → rapid reorganization → attack,

II. attack → defense,

III. defense → withdrawal by order to a quieter sector.

The reader can probably appreciate the difficulties in quantifying such factors.

4. Equivalently, for cases of no replacements and no withdrawals such as we consider here, we may consider the casualty level or casualty fraction. See HELMBOLD [10, p. 7] for a discussion of the case with replacements.

5. The selection of these three factors is motivated by the conclusions reached by D. CLARK [5, p. 3 and also p. 34]. One might add the unit's tactical posture (e.g. attacking, defending in a prepared position, defending in a "hasty defense," etc.) to this list.

6. Method B is apparently due to HELMBOLD [10].

7. In practice this restriction is not as serious as it may at first seem: much more general victory-prediction conditions have so far been obtained by Method B than by Method A. The reason for this situation is that the expression for a force level as a function of time may be very, very complicated (and not expressible in terms of "elementary" functions).

8. The requirement that t_{BP}^{X} be finite is absolutely necessary as an example given in Section 3.6 below shows (see also Footnote 12).

A fourth category, defense → collapse, apparently had to be discarded because of data-base limitations. Thus, one might also call a "breakpoint" a "transition point" in the activities of a combat unit. Such details are apparently not considered in most current combat models whether they be simulations or firepower-score models.

3. D. CLARK [5] considered the following variables in her study of breakpoints:

   (V1) casualties and net casualties (expressed as a percent) on day of breakpoint,

   (V2) cumulative casualties and cumulative net casualties (again, as a percent)

   (a) for day of breakpoint plus two preceding days,

   (b) from start of engagement to breakpoint.

She also briefly considered (subjectively) the possible effects of the following eleven variables:

(1) condition of troops at beginning of engagement,

(2) unusual environmental stresses,

(3) the imperative of the assigned mission,

(4) morale,

(5) leadership,

(6) tactical plan,

(7) reconnaissance,

(8) enemy opposition,

(9) fire support and reinforcement,

(10) logistical support,

(11) communications.

9. Although not explicitly stated in D. CLARK's [5] study, a random-breakpoint model is suggested by her data which showed wide variation in the casualty percentage at which a unit became combat ineffective.

10. Usually, such a break curve relates the probability that a force discontinues the engagement to the casualty fraction (see, for example, HELMBOLD [10]). In cases of no replacements and withdrawals such as the one at hand, this is, of course, equivalent to plotting the probability against the force-level fraction, e.g. x/x_0. For our purposes here, it is more convenient to take the force-level fraction as the independent variable.

11. Unlike our Breakpoint Hypothesis, HELMBOLD [10, p. 7] assumes that the break curves (i.e. breakpoint distributions) are the same for all battles, "irrespective of the size of forces involved or when, where, by whom, or with what the battle was fought." His Hypotheses A and B correspond to our Breakpoint Hypothesis.

12. This is the example that was referred to in Footnote 8 above. Moreover, these equations are a special case of quasi-autonomous equations discussed in Section 6.4 below (see also TAYLOR and BROWN [21, Note 4]).

13. B. O. KOOPMAN (see MORSE and KIMBALL [14, p. 65]) apparently first observed that variable-coefficient equations for a $F|F$ attrition process with attrition-rate coefficients of the form (3.6.8) yield such a square law. This result was apparently later independently discovered by H. K. WEISS [23, p. 88] in a different modelling context.

14. B. O. KOOPMAN (see MORSE and KIMBALL [14, pp. 65-67]) apparently first observed the important result that for a constant ratio of attrition-rate coefficients the, for example, X force level as a function of time, i.e. $x(t)$, takes a form no more complicated than that for constant coefficients: namely, the result (3.6.13) is the same as (2.2.9) except for a transformation of the time scale. Subsequently, ISAACS [12, pp. 327-328], FARRELL [4, pp. 180-184], and TAYLOR [18, Appendix D; 19] have inadvertently rediscovered this result in different modelling contexts (see also TAYLOR [20]). Solutions for special cases of (3.6.8) had been given earlier by BONDER [3].

15. The casualty-fraction conditional distributions have been used in HELMBOLD's [10] empirical investigation of breakpoint hypotheses.

16. Here we have made use of the fact that $P[(F_c^X)_f > p] = P[(F_c^X)_f \geq p]$, since $(F_c^X)_f$ is a continuously distributed r.v.

17. It should be noted that for (3.7.21) to hold we must have $s = 1$ when $t = 1$. Furthermore, the results given in Table 3.II apply for continuously distributed breakpoints. It does not appear that (3.7.21) ever holds exactly in practice for continuously distributed breakpoints, although the results given in Table 3.II are apparently usually excellent approximations to the exact results.

18. HELMBOLD [10, p. 79] has considered (in our notation), for example the d.f.

$$F_X(s) = \begin{cases} 1 - e^{-\lambda_X s} & \text{for } 0 \le s < 1 , \\ 1 & \text{for } s = 1 , \end{cases}$$

and consequently our results are not directly comparable to his. We have chosen to "rescale" the distribution $1 - e^{-\lambda_X s}$ by the factor $1 - e^{-\lambda_X}$, i.e. set $F_X(s) = (1 - e^{-\lambda_X s})/(1 - e^{-\lambda_X})$ as given by (3.8.17), since we felt that results could consequently be more readily compared with those of H. K. WEISS [25].

19. These approximations are not identified as such in WEISS's [25] paper, however. Exact results for P_Y, $P[(F_c^X)_f \le p | X \text{ wins}]$, etc. for distribution functions of the type discussed in Footnote 17 have been given by HELMBOLD [10, pp. 78-82]. Furthermore, WEISS [25] found that, for example, the functional form

$$\hat{\bar{F}}_X(s) = e^{-ks^3} ,$$

gave a good fit to combat data from the U. S. Civil War. He also found that $k > 100$ so that such an approximate complementary d.f. and an approximation like (3.8.22) do indeed yield results extremely close to the exact ones.

20. The historical evidence about the fraction of soldiers who never fire their weapons reported by S. L. A. MARSHALL [13] supports this hypothesis.

21. HELMBOLD [10, pp. 68-69] has shown how such a casualty-fraction break-point distribution may be generated from the conditional probability that a side will fight at least to a specified casualty fraction $(f + h)$ given that it has fought to a given casualty fraction f, i.e.

$$P\left[\begin{array}{l|l}\text{side will fight at least} & \text{side has fought to} \\ \text{to casualty fraction } (f+h) & \text{casualty fraction } f\end{array}\right] = X(h,f) \ .$$

Thus, a break curve may be developed from a continuous model of decision behavior in which a side observes his fractional loss and then decides a MARKOVIAN fashion whether or not continue fighting.

22. Thus, the normalized exchange ratio (or fractional-loss ratio) Γ, defined by $\Gamma = f_c^X/f_c^Y$, is a random variable, which we may consider to be realized (i.e. $\Gamma = \gamma$) before a given battle. This aspect was not explicitly stated by WEISS [25], and subsequently HELMBOLD [10, pp. 50-58] has given the model a more careful examination. Our discussion here, however, follows WEISS [25].

23. Technically speaking, the STIELTJES integral (3.8.6) does not exist when $\gamma = 1$, since both integrand and integrator are discontinuous at the same point (<u>see</u> Appendix B for a further discussion).

24. Strictly speaking, the probability given by (3.12.11) is a conditional probability depending on the realization of the random variable Γ, i.e. $P[Y \text{ will win} | \Gamma = \gamma]$. Then, the unconditional probability that Y will win is given by

$$P_Y = \int_0^\infty P[Y \text{ will win} | \Gamma = \gamma] \, f_\Gamma(\gamma) \, d\gamma.$$

This point is not explicitly stated in WEISS's paper [25] in which $P[Y \text{ will win} | \Gamma = \gamma]$ as given by (3.12.11) would be imprecisely denoted as P_Y. However, to make WEISS's paper [25] accessible to the reader, we have nevertheless chosen to overlook the fact that, for example, (3.12.11) and (3.12.18) are actually conditional results (<u>see</u> HELMBOLD [10, pp. 50-51] for a later and further discussion of this point).

25. Strictly speaking, these results are conditioned on a given realization of the random variable Γ. WEISS does not note this point, and the expressions (3.13.9) and (3.13.10) are the ones actually given by him in [25]. <u>See</u> also Footnote 24 above.

26. HELMBOLD's [10] statement of a general principle for developing victory-prediction conditions (<u>see</u>, for example, our Proposition 3.5.1) is incomplete, however. In order for it to be entirely correct, an addditional condition must be added, namely (for the situation considered by Proposition 3.5.1) that t_{BP}^X must be finite.

27. Actually, one cannot "prove" that HELMBOLD's [10] breakpoint hypothesis is true, but one may be able to show that the hypothesis is at variance with available empirical evidence (and hence untenable).

REFERENCES for Chapter 3

1. J. Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York, 1966.

2. G. Bodart, Militar-Historische Kriegs-Lexicon, (1618-1905), C. W. Stern, Vienna and Leipzig, 1908.

3. S. Bonder, "Combat Model," Chapter 2 in The Tank Weapon System, Report No. RF 573 AR 64-1 (U), Systems Research Group, The Ohio State University, Columbus, Ohio, June 1964 (AD 447 494).

4. S. Bonder and R. Farrell (Editors), "Development of Models for Defense Systems Planning," Report No. SRL 2147 TR 70-2 (U), Systems Research Laboratory, The University of Michigan, Ann Arbor, Michigan, September 1970.

5. D. Clark, "Casualties as a Measure of the Loss of Combat Effectiveness of an Infantry Battalion," ORO-T-289, Operations Research Office, The Johns Hopkins University, Chevy Chase, Maryland, August 1954 (AD 59 384).

6. T. N. Dupuy, "Predicting the Element of Surprise in Combat," Army 27, No. 5, 44-51 (1977).

7. General Research Corporation, "A Hierarchy of Combat Analysis Models," McLean, Virginia, January 1973.

8. R. Helmbold, "Historical Data and Lanchester's Theory of Combat," CORG-SP-128, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, July 1961 (AD 480 975).

9. R. Helmbold, "Historical Data and Lanchester's Theory of Combat, Part II," CORG-SP-190, Combat Operations Research Group, Technical Operations, Inc., Fort Belvoir, Virginia, August 1964 (AD 480 109).

10. R. Helmbold, "Decision in Battle: Breakpoint Hypotheses and Engagement Termination Data," R-772-PR, The RAND Corporation, Santa Monica, California, June 1971.

11. Historical Evaluation and Research Organization (HERO), "The Fundamentals of Land Combat for Developing Computer Simulation Models of Ground and Air-Ground Warfare," unpublished seminar notes, Dunn Loring, Virginia, 1976.

12. R. Isaacs, Differential Games, John Wiley, New York, 1965.

13. S. L. A. Marshall, Men Against Fire, William Morrow & Company, New York, 1947.

14. P. Morse and G. Kimball, Methods of Operations Research, The M.I.T. Press, Cambridge, Massachusetts, 1951.

15. R. Reddock, "Lanchester Combat Models with Suppressive Fire and/or Unit Disintegration," M.S. Thesis in Operations Research, Naval Postgraduate School, Monterey, California, March 1973 (AD 761 759).

16. W. Schmieman, "The Use of Lanchester-Type Equations in the Analysis of Past Military Engagements," Ph.D. Thesis, Georgia Institute of Technology, Atlanta, Georgia, August 1967.

17. S. Spring and S. Miller, "FAST-VAL: Relationships Among Casualties, Suppression, and the Performance of Company-Size Units," RM-6268-PR, The RAND Corporation, Santa Monica, California, March 1970.

18. J. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part I," Tech. Report NPS55TW70062A, Naval Postgraduate School, Monterey, California, June 1970 (AD 717 577).

19. J. Taylor, "A Note on the Solution to Lanchester-Type Equations with Variable Coefficients," Opns. Res. 19, 709-712 (1971).

20. J. Taylor, "Comments on 'A Note on the Solution to Lanchester-Type Equations with Variable Coefficients'," Opns. Res. 20, 1194-1195 (1972).

21. J. Taylor and G. Brown, "Canonical Methods in the Solution of Variable-Coefficient Lanchester-Type Equations of Modern Warfare," Opns. Res. 24, 44-69 (1976).

22. H. K. Weiss, "Requirements for a Theory of Combat," Memorandum Report No. 667, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, April 1953 (AD 13 717).

23. H. K. Weiss, "Lanchester-Type Models of Warfare," pp. 82-98 in Proc. First International Conference on Operational Research, M. Davies, R. Eddison, and T. Page (Editors), Operations Research Society of America, Baltimore, Maryland, 1957.

24. H. K. Weiss, "Stochastic Models for the Duration and Magnitude of a 'Deadly Quarrel'," Opns. Res. 11, 101-121 (1963).

25. H. K. Weiss, "Combat Models and Historical Data: The U.S. Civil War," Opns. Res. 14, 759-790 (1966).

26. D. Willard, "Lanchester as Force in History: An Analysis of Land Battles of the Years 1618-1905," RAC-TP-74, Research Analysis Corporation, Bethesda, Maryland, November 1962.

APPENDIX B: THE PROBABILITY THAT ONE RANDOM VARIABLE IS LESS THAN ANOTHER

1. Introduction.

As we have discussed in Section 3.3 above, many times a military operations analyst would like to have battle-outcome-prediction conditions for various types of tactical engagements. If the combat model of such a tactical situation is simple enough, then one may indeed be able to develop explicit battle-outcome-prediction conditions. Such analytical results are very useful for developing insights into the dynamics of combat. When there are random effects in the combat model, developing such battle-outcome-prediction conditions usually involves, in one way or another, determining the probability that one random variable is less than another that is statistically independent of the first.

One such case that we considered in Chapter 3 was that of determining the probability of winning a battle modelled with deterministic attrition and random breakpoints. Another is that of determining the outcome of a duel between two individual weapon systems, *cf*. the theory of stochastic duals [1]. In the latter case, the probability of winning the duel is equal to the probability of the duelist having the smaller of the two times to kill a passive target. In fact, determining the distribution of combat outcomes for almost any stochastic combat model will invariably involve some type of probability that one random variable is less than another independent one. Although this is certainly not the most general case possible, it does correspond to the only one considered in the literature, and consequently for our present purposes it suffices to consider the case of two independent random variables.

2. Basic Theory.

Consider two independent random variables, denoted as S and T.

QUESTION: What is the probability that S is less than T, i.e. $P[S < T]$?

Before answering this question, let us introduce some necessary notation. We denote the distribution function (d.f.) of the random variable (r.v.) S, also frequently called the cumulative distribution function (c.d.f.), as $F_S(s)$, i.e.

$$F_S(s) = P[S \leq s] = \int_{-\infty}^{s} dF_S(\sigma) ,$$

and similarly for $F_T(t)$. When the random variable S is continuous, i.e. $F_S(s)$ is a continuous function, then the d.f. may be expressed in terms of a probability density function, denoted as $f_S(s)$, i.e.

$$F_S(s) = \int_{-\infty}^{s} f_S(\sigma)\, d\sigma \quad .$$

Also, we will denote the corresponding complementary d.f. as $\overline{F}_S(s)$, i.e. $\overline{F}_S(s) = P[S > s] = 1 - F_S(s)$. When S is continuous, then

$$\overline{F}_S(s) = \int_{s}^{\infty} f_S(\sigma)\, d\sigma \quad .$$

To develop an expression for $P[S < T]$, we first consider the probability that the r.v. T is greater than a given realization s of the independent r.v. S. Hence, we consider (see Figure B.1)

$$P[s < T] = 1 - F_T(s) = \overline{F}_T(s) = P[S < T|S = s] , \tag{B.1}$$

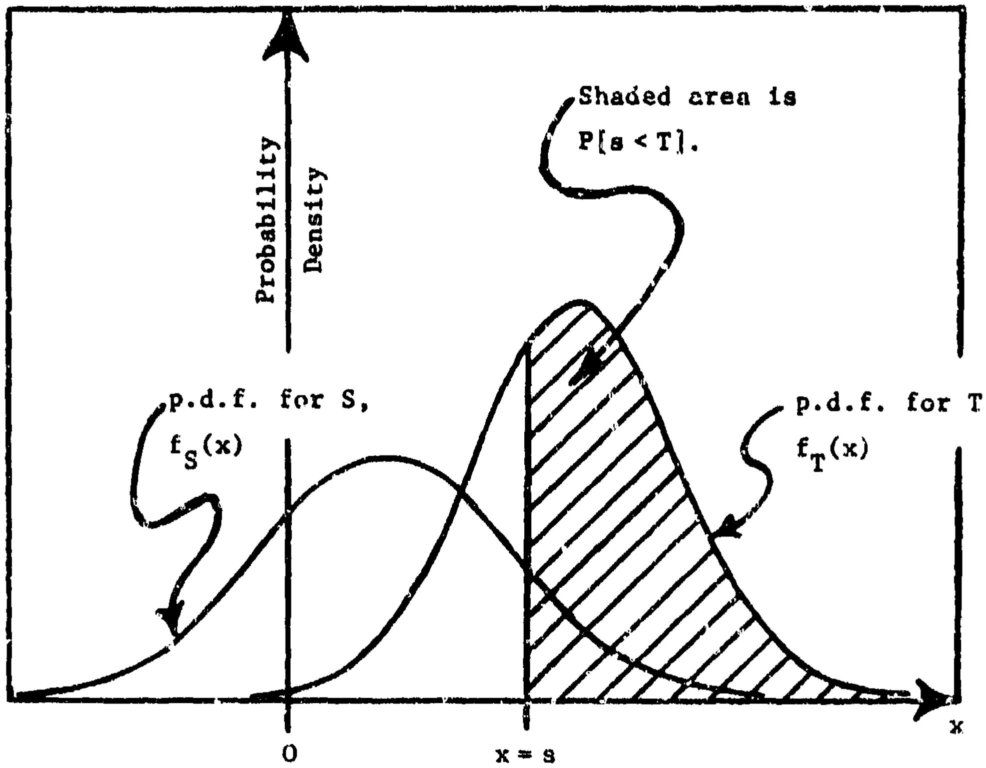


Figure B.1. The probability that one random variable is greater than a given realization of another independent random variable,

$P[s < T] = P[S < T \mid S = s]$.

the last equality on the right-hand side of (B.1) holding by the assumed independence of S and T. Then, unconditioning [i.e. "averaging" the above conditional probability (B.1) over "all possible values of x"], we have

$$P[S < T] = \sum_{\substack{\text{all possible} \\ \text{values of } s}} P[S < T | S = s] \cdot P[s < S \leq s + ds]$$

$$= \int_{-\infty}^{\infty} \overline{F}_T(s) \; dF_S(s) \; .$$

Thus,

$$P[S < T] = \int_{-\infty}^{\infty} \overline{F}_T(s) \; dF_S(s) \; , \tag{B.2}$$

which holds for any two independent random variables. When the random variable S is continuous, the above becomes

$$P[S < T] = \int_{-\infty}^{\infty} \overline{F}_T(s) \; f_S(s) \; ds \; . \tag{B.3}$$

One can also show (either by integrating (B.2) by parts or by using first principles) that

$$P[S < T] = \int_{-\infty}^{\infty} F_S(t) \; dF_T(t) \; . \tag{B.4}$$

Thus, we observe that when one of the r.v. is continuous, then $P[S < T] = P[S \leq T]$. In summary, we have found that for two independent random variables S and T

$$P[S < T] = \int_{-\infty}^{\infty} \overline{F}_T(s) \; dF_S(s) = \int_{-\infty}^{\infty} F_S(t) \; dF_T(t) \; . \tag{B.5}$$

Another case of considerable interest, which is a generalization of the above, is that in which we have a function of the random variable T, denoted here as $\psi(T)$. Again, we assume that the random variables S and T are independent. We now seek to determine the probability that $S < \psi(T)$, namely $P[S < \psi(T)]$. We assume that ψ has a well-defined inverse function, denoted as ψ^{-1}. Replacing T by $\psi(T)$ in the above development, we have

$$P[S < \psi(T)] = P[\psi^{-1}(S) < T]$$

$$= \sum_{\substack{\text{all possible} \\ \text{values of } s}} P[\psi^{-1}(S) < T | S = s] \cdot P[s < S \leq s + ds] ,$$

or

$$P[S < \psi(T)] = \int_{-\infty}^{\infty} \overline{F}_T(\psi^{-1}(s))\ dF_S(s) , \tag{B.6}$$

by virtue of the assumed independence of S and T. Alternatively, we may write

$$P[S < \psi(T)] = \sum_{\substack{\text{all possible} \\ \text{values of } t}} P[S < \psi(T) | T = t] \cdot P[t < T \leq t + dt] ,$$

or

$$P[S < \psi(T)] = \int_{-\infty}^{\infty} F_S(\psi(t))\ dF_T(t) , \tag{B.7}$$

since $F_T(t)$ is continuous from the right, and the integrand $P[S < \psi(T)|T = t]$ as a function of t must be continuous from the left at points of discontinuity of $F_T(t)$ (i.e. $P[S < \psi(T)|T = t] = P[S \leq \psi(T)|T = t]$) in order for the STIELTJES integral to exist (<u>see</u> APOSTOL [2, pp. 212-213]).

Thus, for $s = \psi(t)$ such that ψ^{-1} is well defined, we have shown that

$$P[S < \psi(T)] = \int_{-\infty}^{\infty} \overline{F}_T(\psi^{-1}(s))\ dF_S(s) = \int_{-\infty}^{\infty} F_S(\psi(t))\ dF_T(t) . \tag{B.8}$$

3. Probability of Winning Battle with Deterministic Attrition and Random Breakpoints.

In this section we apply the above basic theory to develop equations (3.7.5) and (3.7.8) of Chapter 3. Consider combat between two homogeneous forces, denoted as X and Y. As in Chapter 3, let $F_X(s)$ denote the d.f. for X's casualty-fraction breakpoint (a r.v.) $(F_c^X)_{BP}$, i.e.

$$F_X(s) = P[(F_c^X)_{BP} \leq s] ,$$

and similarly for $F_Y(t)$. For convenience let us denote $(F_c^X)_{BP}$ simply as S and $(F_c^Y)_{BP}$ as T. Then, equation (3.7.4) of Chapter 3 reads

$$P_Y = P[Y \text{ will win}] = P[S < \psi(T)] , \tag{B.9}$$

where $\psi(t)$ denotes the "truncated" φ-function, i.e. $\psi(t) = \text{Minimum}[\varphi(t),1]$, and φ is the strictly increasing function that relates the combatants' casualty fractions, i.e. $f_c^X = \varphi(f_c^Y)$. As discussed in Section 3.7, ψ^{-1} is then well defined. Assuming that $(F_c^X)_{BP}$ and $(F_c^Y)_{BP}$ are independent, we can combine (B.8) and (B.9) to find that

$$P_Y = \int_0^1 \overline{F}_Y(\psi^{-1}(s))\, dF_X(s) = \int_0^1 F_X(\psi(t))\, dF_Y(t) , \tag{B.10}$$

since, for example, $dF_X(s) = 0$ for $s < 0$ or $s > 1$. Equation (B.10) appears in Chapter 3 as (3.7.5) and (3.7.8).

4. Probability That Next Casualty is of a Given Type in Battle with Randomly Occurring Casulaties (Includes Stochastic Duels).

Other applications of the above basic theory occur in the theory of stochastic duels (see ANCKER [1]) and also for MARKOV-chain versions of LANCHESTER-type combat models (see BROWN [3], SMITH [4], or Chapter 4). We will consider the special case in which the times between casualties are exponentially distributed.

Again, we consider combat between two homogeneous forces, denoted as X and Y. Let S denote the time to the next X casualty and T denote the time to the next Y casualty. Then

$$P[\text{next casualty is an } X \text{ casualty}] = P[S < T] . \tag{B.11}$$

We will assume that S and T are independent. This assumption is the usual one made in the theory of stochastic duels and is also well known to hold for such MARKOV-chain attrition models. If the times between casualties are exponentially distributed, i.e. $S \sim e(\lambda_S)$ and $T \sim e(\lambda_T)$ where $S \sim e(\lambda_S)$ means that "S is exponentially distributed with parameter λ_S" (namely, $P[s < S \leq s + ds] = \lambda_S e^{-\lambda_S s} ds$ for $s \geq 0$), then it follows that for $s \geq 0$

$$\overline{F}_T(s) = \int_s^\infty \lambda_T e^{-\lambda_T t} dt = e^{-\lambda_T s} ,$$

and

$$f_S(s) = \lambda_S e^{-\lambda_S s} .$$

Hence, by (B.3) we have

$$P[S < T] = \int_0^\infty \overline{F}_T(s)\ f_S(s)\ ds = \int_0^\infty \lambda_S\ e^{-(\lambda_S+\lambda_T)s}\ ds$$

$$= -\left(\frac{\lambda_S}{\lambda_S + \lambda_T}\right) e^{-(\lambda_S+\lambda_T)s}\Bigg|_{s=0}^{s=+\infty} .$$

Thus, for times between events (i.e. casualties) being exponentially distributed, we have

$$P[\text{next casualty is an } X \text{ casualty}] = \frac{\lambda_S}{\lambda_S + \lambda_T}, \tag{B.12}$$

where λ_S denotes X's casualty rate and λ_T denotes Y's casualty rate.

REFERENCES for Appendix B

1. C. J. ANCKER, "The Status of Developments in the Theory of Stochastic Duels --II," Opns. Res. 15, 388-406 (1967).

2. T. M. APOSTOL, "Mathematical Analysis, Addison-Wesley, Reading, Massachusetts, 1957.

3. R. BROWN, "Theory of Combat: The Probability of Winning," Opns. Res. 11, 418-425 (1963).

4. D. SMITH, "The Probability Distribution of the Number of Survivors in a Two-sided Combat Situation," Operational Res. Quart. 16, 429-437 (1965).

Chapter 4. STOCHASTIC LANCHESTER-TYPE COMBAT MODELS

4.1. Introduction.

Combat is anything but a deterministic process. Military history gives us innumerable examples of random effects (or just plain luck) playing a major role in warfare. Why, then, have we been considering only deterministic combat formulations so far in this book? The reason is quite simple: although there is no mathematical difficulty in formulating stochastic LANCHESTER-type combat models, it is very difficult to obtain information about the dynamical behavior of the model (e.g. to answer questions like questions (Q1) through (Q7) posed in Section 2.2 above). Furthermore, although random effects may at first appear to be significant in combat, we feel that for many combat situations the "first order" (or primary) nature of the combat dynamics may be observed in a deterministic model, with the random effects usually being secondary and qualifying the "first order" behavior observed in the deterministic model.

Many times, however, a deterministic model may be a very bad representation of random combat phenomena and may yield misleading insights into the dynamics of combat. Thus, there is a greatly enhanced value to deterministic formulations of the combat attrition process when properly interpreted within the framework of more comprehensive stochastic models (including Monte Carlo simulations). In fact, the "empirical" estimation of LANCHESTER attrition-rate coefficients (for a deterministic combat model) from high-resolution Monte Carlo simulation or field experimentation data can be based on a stochastic model of the combat attrition process.

LANCHESTER himself had in the back of his mind that the true nature of combat was stochastic and that his simple deterministic differential-equation models were only approximations (valid to some extent for large

numbers of combatants) to the average, or mean, course of battle. Considering the decay of the force levels in combat, LANCHESTER [56, p. 422, column 3] said,

> "Since the forces actually consist of a finite number of units (instead of an infinite number of infinitesimal units), the end of the curve must show discontinuity, and break off abruptly when the last man is reached; the law based on averages evidently does not hold rigidly when the numbers become small."

Thus, LANCHESTER stated that his deterministic differential-equation models were "based on averages," implying an underlying stochastic process. Furthermore, as the above quotation shows, he realized that such deterministic differential equations may yield good approximations to the mean of the underlying stochastic process only when the force sizes are "large." We should therefore view deterministic LANCHESTER-type equations as representing (in some sense) the mean or expected course of battle and view them somewhat skeptically for "small" numbers of combatants.

During World War II, B. O. KOOPMAN (see MORSE and KIMBALL [65, pp. 67-71] extended LANCHESTER's [56] results and developed stochastic versions of LANCHESTER's original models, which we have considered above in Chapter 2. Many other workers have subsequently considered a stochastic analysis of combat attrition.[1]

There are many random factors present on the battlefield in combat. We can improve the realism of LANCHESTER-type models by including random variations in the following:

(R1) the attrition-rate coefficients (they may be random variables that are realized before the engagement begins),

(R2) the enemy's initial force level (the exact numerical strength of the enemy is usually unknown at the beginning of a battle for it, i.e. random initial conditions for the enemy),

(R3) the breakpoints (i.e. random stopping mechanism),

and (R4) the occurrence of casualties (but at specified rates).

In this book, however, we will consider only random variations in the occurrence of casualties over time. Moreover, this is apparently the only source of random effects that has so far been considered in the combat-modelling literature.

As we will see in the next section, it is a simple task to formulate stochastic versions of any particular deterministic LANCHESTER-type model. However, there is usually at least an order of magnitude more difficulty in analytically extracting information from such a stochastic model than in extracting the analogous information from a corresponding deterministic model. Fortunately, the behavior of the deterministic model is a good guide for studying and describing the behavior of a stochastic version[2] of this model. Thus, we should view the deterministic results as a benchmark, a point of departure for discussing stochastic results.

In other words, we should ask ourselves, "How do random fluctuations in the occurrence of casualties modify the deterministic results?" This

appears to be a reasonable approach for studying probabilistic combat dynamics, and it is the one that we will follow. Accordingly, in Section 4.4 below, we will discuss what information should be obtained from the model. Basically, we seek answers to probabilistic versions of questions (Q1) through (Q7) posed in Section 2.2 above. Typical quantities of interest are now expressed in probabilistic terms, e.g.

(1) the probability of winning,

(2) the average force levels as a function of time.

As G. CLARK [16] has emphasized, the deterministic and stochastic models are related, since the deterministic model should represent the mean or expected course of battle. Moreover, an analysis of the stochastic formulations is useful in understanding the impact of random fluctuations in the occurrences of casualties upon the outcome of battle. It is also useful for interpreting the deterministic model in the sense that it can reveal how accurately the deterministic model approximates the expected outcome of a more general stochastic combat process.

4.2. Probabilistic Dynamic Models.

In this chapter we will study a probabilistic version (in which casualties occur randomly over time) of the LANCHESTER-type equations for combat between two homogeneous forces[3]. Extension to combat between heterogeneous forces follows along obvious lines.

We begin by first considering a deterministic combat-attrition model and then develop its stochastic analogue in which there are random fluctuations in the occurrences of casualties over time. Let us therefore consider combat between two homogeneous forces described by the following deterministic LANCHESTER-type equations for $x,y > 0$ [the first equation, for example, becomes $dx/dt = 0$ for $x = 0$]

$$\begin{cases} \dfrac{dx}{dt} = -\,G(t,x,y) & \text{with } x(0) = x_0, \\ \dfrac{dy}{dt} = -\,H(t,x,y) & \text{with } y(0) = y_0, \end{cases} \tag{4.2.1}$$

where $x(t)$ and $y(t)$ denote the X and Y force levels at time t, and G and H denote force-change rates (with a negative force-change rate signifying a net influx of replacements). For simplicity we assume that there are no replacements and withdrawals; and, in this case, G and H are simply casualty rates. This simple combat situation is shown diagrammatically in Figure 4.1.

The above equations (4.2.1) are a deterministic dynamic model of combat between two homogeneous forces. In this model

$$t = \text{"time" parameter}$$
$$x,y = \text{state variables}[4].$$

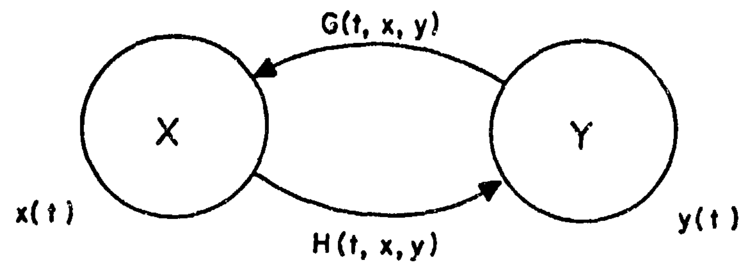


Figure 4.1. Combat between two homogeneous forces with no replacements and no withdrawals.

Both the time parameter and the state variables are taken in (4.2.1) to be nonnegative real numbers. It is intuitively appealing to model time as a continuous variable (i.e. a variable that can take on a continuum of values), although many times it is more convenient to consider the evolution of a dynamic system at only discrete points in time and to consider time as a discrete variable.

In choosing to use a deterministic model such as (4.2.1), we have found it convenient to represent, for example, the integral number of X combatants (i.e. physically the X force level can only be a nonnegative integer) with the real number $x(t)$. In other words, although we know it really isn't true, we consider that the force levels are continuous variables. This is a compromise that one must frequently make in order to use a differential-equation model (which here implies differentiability of the force levels) to represent the evolution of a dynamic system. We intuitively feel that this is a "reasonable" idealization for "large" force levels, and we should bear in mind that all models involve such abstractions (<u>see</u> Section 1.1.2 for further discussion of this point).

Thus, we seek a probabilistic version of the above deterministic LANCHESTER-type model (4.2.1). Such a model is called a <u>MARKOV process</u>, since the probability of any particular future behavior of the process is entirely determined by the present state. More formally, we have the following definition of a MARKOV process (for further details, <u>see</u> PARZEN [69] or KARLIN [44]:

DEFINITION 4.1: A random process is called a <u>MARKOV process</u> when there is no dependence on past history; the future probabilistic behavior of the system depends solely on its current state.

In fact, the concept of a MARKOV process was developed to be a probabilistic analogue of a deterministic process modelled by differential equations like (4.2.1).[5]

Depending on how we model "time" and the system state, there are different types of MARKOV processes which we can use to model a dynamic system. It is therefore convenient to classify MARKOV processes according to [69, p. 188]

(C1) the nature of the "time" parameter (whether it is a discrete or a continuous variable),

(C2) the nature of the state space of the process.

Such a classification is more or less the standard one.

Thus, we have a choice of the type of probabilistic version for (4.2.1) to consider. Of all the different types of MARKOV processes shown in Figure 4.2, it is easiest to extract the information in which we are interested (<u>see</u> Section 4.4) from the continuous-parameter MARKOV-chain model, and this model does take into consideration the integer restrictions on the combatants' force levels. Thus, we will consider combat modelled as a continuous-parameter MARKOV chain. [In order to heuristically develop the relationship between the deterministic model (4.2.1) and its MARKOV-process analogue, however, we will also briefly consider combat modelled as a continuous-parameter MARKOV process, i.e. diffusion process.] In fact, since we will assume that no more than one casualty can occur at a time, the MARKOV chain will be of a special type called a <u>birth-death process</u> (actually a pure death process; <u>see</u> KLEINROCK [54, especially p. 25] for further details).

		State Space	
		DISCRETE	CONTINUOUS
"TIME" PARAMETER	DISCRETE	DISCRETE PARAMETER MARKOV CHAIN	DISCRETE PARAMETER MARKOV PROCESS
	CONTINUOUS	CONTINUOUS PARAMETER MARKOV CHAIN	CONTINUOUS PARAMETER MARKOV PROCESS (DIFFUSION PROCESS)

Figure 4.2. Classification of MARKOV processes.

4.3. A MARKOV-Chain Model for LANCHESTER-Type Combat.

Let us therefore consider combat attrition modelled as a continuous-parameter MARKOV chain corresponding to the general deterministic LANCHESTER-type homogeneous attrition process (4.2.1). Thus, time varies continuously in our model, but the number of live (i.e. effective) combatants (assumed to be homogeneous) on each side is a nonnegative integer. Furthermore, the number of live combatants on each side is a random variable, since we are now assuming that casualties occur randomly over time.

Following our notational convention of denoting random variables by upper-case letters and their realizations by the corresponding lower-case letters, we will denote the X force level at time t, a random variable (frequently abbreviated r.v.), as $M(t)$, with its realization at time t being denoted as m. Corresponding quantities for the Y forces will be denoted as $N(t)$ and n. Initially in the battle (i.e. at $t = 0$) there are m_0 X combatants and n_0 Y combatants with certainty, i.e. with probability one. All these quantities M, N, m, and n are restricted to be nonnegative integers. Hence, the random variable $M(t)$ can take on the value $m = 0,1,2,\ldots, m_0$, each with some positive probability in a fight to the finish; and similarly for $N(t)$. Our simple combat situation is shown diagrammatically in Figure 4.3. Here $G = G(t,m,n)$ denotes the casualty rate of the X force, since we are assuming no replacements and withdrawals (see Section 4.2), and similarly for $H = H(t,m,n)$.

Corresponding to the two deterministic differential equations (4.2.1) is a system of many more differential-difference equations.[6] Furthermore, this system of equations is actually influenced by the engagement-termination model adopted for the battle. For simplicity, we will first consider a fight to the finish (i.e. a fight that lasts until the annihilation of one side or the other), and then we will subsequently consider the equations for a fixed-force-level-

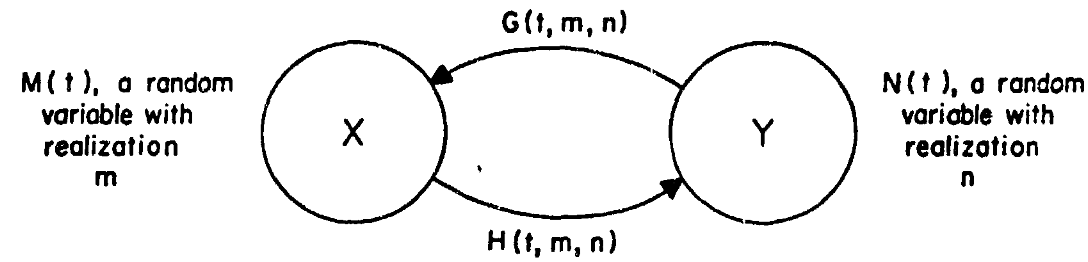


Figure 4.3. Homogeneous-force combat modelled as a continuous-parameter MARKOV chain. Here $M(t)$, a random variable (r.v.), denotes the number of X combatants at time t; and $N(t)$, a r.v., denotes the corresponding number for Y. The combat depicted here is the stochastic analogue of that shown in Figure 4.1 (i.e. combat with no replacements and no withdrawals).

breakpoint battle (see Section 2.8). Thus, we now turn to the development of the equations (here the so-called forward KOLMOGOROV equations[7]) that govern the evolution of the probability distribution for the number of survivors; some people call this vector of state probabilities the state-probability vector.

It is more convenient, however, for us to consider the evolution of individual components of the state-probability vector than to consider the vector itself. Let us therefore denote an individual component of state-probability vector as follows

$$P(t,m,n;0,m_0,n_0) = P\left[\begin{array}{c|c} M(t)=m & M(0)=m_0 \\ & \\ N(t)=n & N(0)=n_0 \end{array}\right] ,$$

where $P[A = a_1]$ denotes the probability that a random event A has the outcome a_1. For notational convenience, we will denote $P(t,m,n;0,m_0,n_0)$ simply as $P(t,m,n)$. In other words, we have

$$P(t,m,n) = P\left[\begin{array}{c|c} M(t)=m & M(0)=m_0 \\ & \\ N(t)=n & N(0)=n_0 \end{array}\right] . \tag{4.3.1}$$

We will now briefly focus on the assumed attrition-rates of our model. In the deterministic model (4.2.1) with no replacements and withdrawals, G is called the attrition rate of the entire X force, since $(-dx/dt) = G(t,x,y)$. In our MARKOV-chain model, the stochastic analogue of (4.2.1), we analogously have

$G(t,m,n)$ = rate of attrition for the entire X force,

$H(t,m,n)$ = rate of attrition for the entire Y force.

Then, as one usually assumes (e.g. see KARLIN [44, p. 189]), we assume that

$$P\begin{bmatrix}\text{one X casualty during interval}\\ \text{of time from } t \text{ to } t+\Delta t\end{bmatrix} = G(t,m,n)\Delta t\ ,$$

(4.3.2)

$$P\begin{bmatrix}\text{one Y casualty during interval}\\ \text{of time from } t \text{ to } t+\Delta t\end{bmatrix} = H(t,m,n)\Delta t\ .$$

We further assume that during any short interval of time of length Δt the probability of more than one casualty (either on the same side or on both sides) is negligible. In mathematical terms we express this assumption as

$$P\begin{bmatrix}\text{more than one casualty during interval}\\ \text{of time from } t \text{ to } t+\Delta t\end{bmatrix} = O\left((\Delta t)^2\right), \qquad (4.3.3)$$

where $O(x)$ denotes dependence on x such that $\lim_{x\to 0} O(x)/x$ = CONSTANT, i.e. $\lim_{x\to 0} O(x^2)/x = 0$.

The battle-termination conditions are the final ingredient to our combat model and are incorporated into the model in the following way. Corresponding to the assumption made for the deterministic model that $dx/dt = 0$ and $dy/dt = 0$ when $x = 0$ or $y = 0$ is the assumption that <u>no more casualties can occur when $m = 0$ or $n = 0$.</u> The quantification of this assumption incorporates the battle-termination conditions into our combat model. Thus, when $m = 0$ or $n = 0$, the combat state is absorbing, and the dynamics of

our stochastic combat model must be modified. We will see how this is done below. Accordingly, we will first consider the case in which $0 < m \leq m_0$ and $0 < n \leq n_0$.

After these preliminaries, we will now turn to the development of the forward KOLMOGOROV equations, which describe the probabilistic evolution of the state of each of our two opposing combat systems. Let us first observe that the initial conditions for the forward KOLMOGOROV equations are given by

$$P(0,m,n) = \begin{cases} 1 & \text{for } m = m_0 \text{ and } n = n_0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.4}$$

since we have assumed that there are initially m_0 X combatants and n_0 Y combatants with certainty. Also, since we are assuming that there are no replacements, it is impossible to have $m > m_0$ or $n > n_0$. Thus, we conclude that

$$P(t,m,n) = 0 \qquad \text{for } m > m_0 \text{ or } n > n_0 . \tag{4.3.5}$$

This result will allow us to simplify the development of the forward KOLMOGOROV equations by allowing battle states (m,n) with $m = m_0$ or $n = n_0$ to be considered as special cases of those for $0 < m \leq m_0$ and $0 < n \leq n_0$. We now consider the development of these basic equations for this latter general case (slightly different developments are required when $m = 0$ or $n = 0$ (see below)).

Thus, for $0 < m \leq m_0$ and $0 < n \leq n_0$, the usual conditional probability arguments (e.g. see FELLER [25, pp. 407-408]) yield

$$
P\begin{bmatrix} M(t+\Delta t) = m \\ N(t+\Delta t) = n \end{bmatrix}
$$

$$
= P\begin{bmatrix} M(t) = m \\ N(t) = n \end{bmatrix} \cdot P\begin{bmatrix} \text{no casualty occurred on either side in} \\ \text{interval of time from } t \text{ to } t+\Delta t \end{bmatrix}
$$

$$
+ P\begin{bmatrix} M(t) = m+1 \\ N(t) = n \end{bmatrix} \cdot P\begin{bmatrix} \text{one } X \text{ casualty occurred in interval} \\ \text{of time from } t \text{ to } t+\Delta t \end{bmatrix}
$$

$$
+ P\begin{bmatrix} M(t) = m \\ N(t) = n+1 \end{bmatrix} \cdot P\begin{bmatrix} \text{one } Y \text{ casualty occurred in interval} \\ \text{of time from } t \text{ to } t+\Delta t \end{bmatrix}
$$

$$
+ P\begin{bmatrix} \text{combatants in some} \\ \text{other state at } t \end{bmatrix} \cdot P\begin{bmatrix} \text{more than one casualty occurred} \\ \text{in interval of time from } t \text{ to } t+\Delta t \end{bmatrix} \qquad (4.3.6)
$$

since if we find ourselves (with some probability) at battle state (m,n) at time $t + \Delta t$, then one of the following four mutually exclusive events must have occurred (<u>see</u> Figure 4.4.):

1. we were in battle state (m,n) at time t and no casualty occurred in the time interval $(t, t + \Delta t)$.

2. we were in battle state $(m+1, n)$ at time t and one X casualty occurred in the time interval $(t, t + \Delta t)$,

3. we were in battle state $(m, n+1)$ at time t and one Y casualty occurred in the time interval $(t, t + \Delta t)$,

4. we were in some other battle state at time t and more than one casualty occurred in the time interval $(t, t + \Delta t)$.

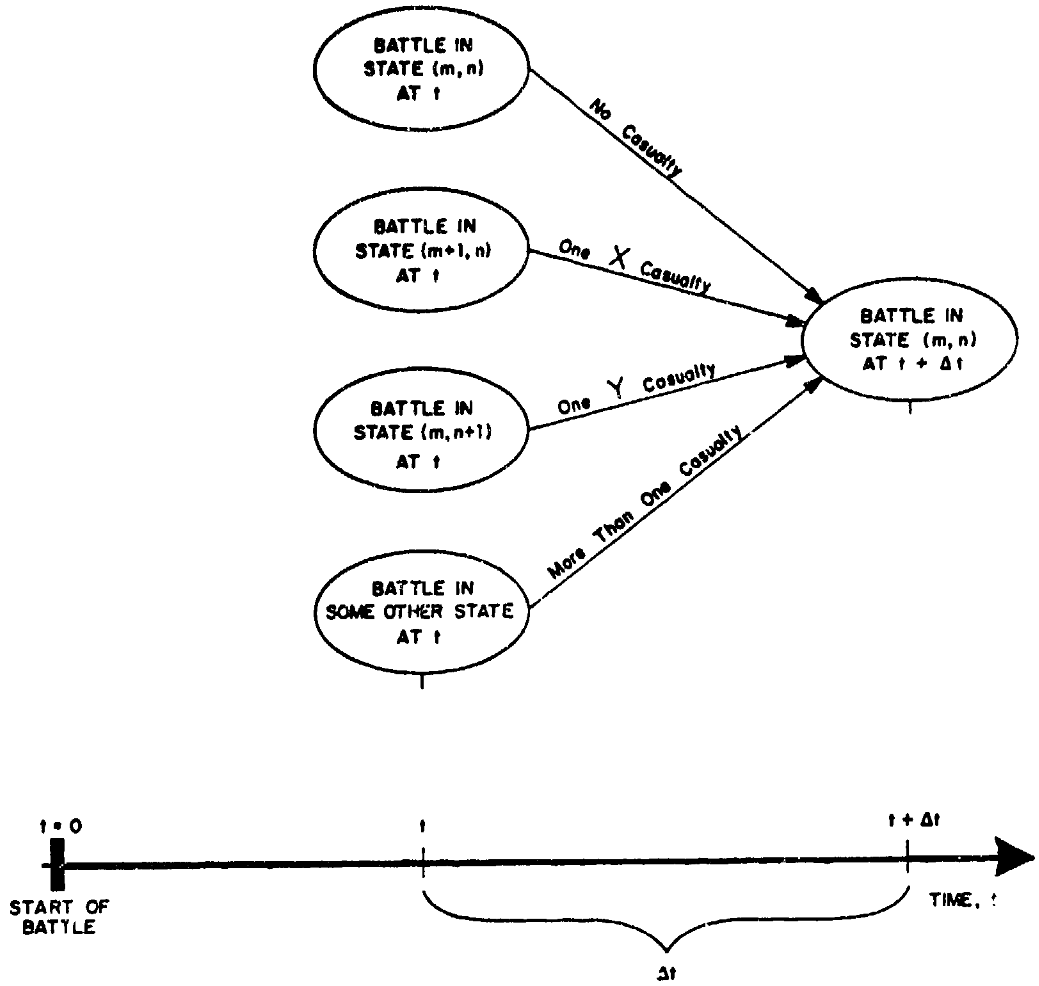


Figure 4.4. Possible battle-state transitions considered in the development of the forward KOLMOGOROV equations (4.3.9).

Here the MARKOV assumption that the future depends only on the present, i.e. it is independent of past history (recall Definition 4.1), allows us to write, for example,

$$P\begin{bmatrix} M(t) = m+1 \text{ and } N(t) = n \text{ and one } X \text{ casualty} \\ \text{occurred in interval of time from } t \text{ to } t + \Delta t \end{bmatrix}$$

$$= P\begin{bmatrix} M(t) = m+1 \\ N(t) = n \end{bmatrix} \cdot P\begin{bmatrix} \text{one } X \text{ casualty occurred in interval} \\ \text{of time from } t \text{ to } t + \Delta t \end{bmatrix},$$

where for notational convenience we have denoted, for example,

$P\begin{bmatrix} M(t) = m \mid M(0) = m_0 \\ N(t) = n \mid N(0) = n_0 \end{bmatrix}$ simply as $P\begin{bmatrix} M(t) = m \\ N(t) = n \end{bmatrix}$. Considering (4.3.2),

we observe that

$$P\begin{bmatrix} \text{no casualty occurred on either side} \\ \text{in interval of time from } t \text{ to } t + \Delta t \end{bmatrix}$$

$$= \{1 - G(t,m,n)\Delta t\}\cdot\{1 - H(t,m,n)\Delta t\}$$

$$= 1 - \{G(t,m,n) + H(t,m,n)\}\Delta t + O\left((\Delta t)^2\right). \tag{4.3.7}$$

Next, ignoring[8] terms $O\left((\Delta t)^2\right)$, we obtain from (4.3.6)

$$P(t + \Delta t,m,n) = P(t,m,n)\{1 - G(t,m,n)\Delta t - H(t,m,n)\Delta t\} + P(t,m+1,n)\, G(t,m+1,n)\Delta t$$
$$+ P(t,m,n+1)\, H(t,m,n+1)\Delta t, \tag{4.3.8}$$

where we have made use of (4.3.1), (4.3.2), (4.3.5), and (4.3.7). Writing (4.3.8) as

$$\frac{P(t + \Delta t,m,n) - P(t,m,n)}{\Delta t}$$

$$= G(t,m+1,n)\ P(t,m+1,n) + H(t,m,n+1)\ P(t,m,n+1)$$
$$- \{G(t,m,n) + H(t,m,n)\}\ P(t,m,n) \ ,$$

and letting $\Delta t \to 0$, we obtain the <u>forward KOLMOGOROV equations</u> for the (forward) probabilistic evolution of the probability distribution over the numbers of survivors on both sides in our stochastic battle. Thus, we obtain for $0 < m \leq m_0$ and $0 < n \leq n_0$

$$\frac{dP}{dt}(t,m,n) = G(t,m+1,n)\ P(t,m+1,n) + H(t,m,n+1)\ P(t,m,n+1)$$
$$- \{G(t,m,n) + H(t,m,n)\}\ P(t,m,n), \tag{4.3.9}$$

with initial conditions (4.3.4) at $t = 0$. Here the reader should keep in mind the result (4.3.5). This LANCHESTER-type stochastic process[9] is called (appropriately enough) a "pure death" process, since we can only have "downward" state transitions from $(m+1,n)$ to (m,n) or from $(m,n+1)$ to (m,n). However, the above forward KOLMOGOROV equations only apply for m and $n > 0$. On the boundary of the state space they take a slightly different form.

On the boundary of the state space where $m = 0$ or $n = 0$, no more casualties can occur, and the above developments must be slightly modified. Thus, for $m = 0$ and $0 < n \leq n_0$

$$P\begin{bmatrix} M(t+\Delta t) = 0 \\ N(t+\Delta t) = n \end{bmatrix}$$

$$= P\begin{bmatrix} M(t) = 0 \\ N(t) = n \end{bmatrix} + P\begin{bmatrix} M(t) = 1 \\ N(t) = n \end{bmatrix} \cdot P\begin{bmatrix} \text{one } X \text{ casualty occurred in} \\ \text{interval of time from } t \text{ to } t+\Delta t \end{bmatrix}$$

$$+ P\begin{bmatrix} \text{combatants in some} \\ \text{other state at } t \end{bmatrix} \cdot P\begin{bmatrix} \text{more than one casualty occurred in} \\ \text{interval of time from } t \text{ to } t+\Delta t \end{bmatrix},$$

since if we find ourselves (with some probability) at battle state (m,n) at time $t + \Delta t$, then one of the following three mutually exclusive events must have occurred

1. we were in battle state $(0,n)$ at time t and with certainty no casualties occurred in the time interval $(t, t + \Delta t)$,
2. we were in battle state $(1,n)$ at time t and one X casualty occurred in the time interval $(t, t + \Delta t)$,
3. we were in some other battle state at time t and more than one casualty occurred in the time interval $(t, t + \Delta t)$.

Here the first term on the right-hand side of the above equation of total probability is a reflection of the battle-termination model, i.e. fixed-force-level-breakpoint battle. The usual arguments based on passing to the limit as $\Delta t \to 0$ now yield that for $m = 0$ and $0 < n \leq n_0$

$$\frac{dP}{dt}(t,0,n) = G(t,1,n)\ P(t,1,n)\ . \tag{4.3.10}$$

Similarly, we find that for $0 < m \leq m_0$ and $n = 0$

$$\frac{dP}{dt}(t,m,0) = H(t,m,1)\ P(t,m,1)\ . \tag{4.3.11}$$

Finally, the assumption that it is impossible for more than one casualty at a time to occur, cf. (4.3.3), yields that it is impossible to reach the state (0,0), and hence

$$P(t,0,0) = 0\ . \tag{4.3.12}$$

The system of forward KOLMOGOROV equations (4.3.9) through (4.3.12) with initial conditions (4.3.4) is a stochastic version (others are possible; see Footnote 2) of the deterministic combat-attrition model (4.2.1) with battle-termination conditions that dx/dt and $dy/dt = 0$ when either $x = 0$ or $y = 0$. The reader should keep in mind that the result (4.3.5) applies to (4.3.9) so that the forward KOLMOGOROV equations take a special form when $m = m_0$ and $n = n_0$. Consequently, the reader should think of (4.3.9) (and hence the entire system) as an "abbreviated" form of the stochastic combat equations. Thus, although it may sometimes be convenient for one to consider this abbreviated form of the forward KOLMOGOROV equations (4.3.9) through (4.3.12), the reader should note that written out in full the forward KOLMOGOROV equations are

for $m = m_0$ and $n = n_0$

$$\frac{dP}{dt}(t,m_0,n_0) = -\{G(t,m_0,n_0) + H(t,m_0,n_0)\}\ P(t,m_0,n_0), \tag{4.3.13}$$

for $0 < m < m_0$ and $n = n_0$

$$\begin{aligned}\frac{dP}{dt}(t,m,n_0) = {} & G(t,m+1,n_0)\ P(t,m+1,n_0) \\ & - \{G(t,m,n_0) + H(t,m,n_0)\}\ P(t,m,n_0),\end{aligned} \tag{4.3.14}$$

for $m = m_0$ and $0 < n < n_0$

$$\begin{aligned}\frac{dP}{dt}(t,m_0,n) = {} & H(t,m_0,n+1)\ P(t,m_0,n+1) \\ & - \{G(t,m_0,n) + H(t,m_0,n)\}\ P(t,m_0,n),\end{aligned} \tag{4.3.15}$$

for $0 < m < m_0$ and $0 < n < n_0$

$$\begin{aligned}\frac{dP}{dt}(t,m,n) = {} & G(t,m+1,n)\ P(t,m+1,n) + H(t,m,n+1)\ P(t,m,n+1) \\ & -\{G(t,m,n) + H(t,m,n)\}\ P(t,m,n),\end{aligned} \tag{4.3.16}$$

for $m = 0$ and $0 < n \leq n_0$

$$\frac{dP}{dt}(t,0,n) = G(t,1,n)\ P(t,1,n), \tag{4.3.17}$$

for $0 < m \leq m_0$ and $n = 0$

$$\frac{dP}{dt}(t,m,0) = H(t,m,1)\, P(t,m,1), \qquad (4.3.18)$$

and for $m = 0$ and $n = 0$

$$P(t,0,0) = 0 \qquad \text{for all } t \geq 0\ , \qquad (4.3.19)$$

with initial conditions (4.3.4). The reader should observe the symmetry exhibited by the forward equations (4.3.13) through (4.3.15) and (4.3.17) through (4.3.19) on symmetric portions of the state-space boundary where $m = 0$ or m_0 and/or $n = 0$ or n_0.

Let us now summarize the assumptions made for the development of the above equations for this MARKOV-type attrition process:

(A1) the attrition process depends on the current system state and time, but it does not depend on past history,

(A2) $P\begin{bmatrix}\text{one } X \text{ casualty during interval} \\ \text{of time from } t \text{ to } t + \Delta t\end{bmatrix} = G(t,m,n)\Delta t,$

and

$P\begin{bmatrix}\text{one } Y \text{ casualty during interval} \\ \text{of time from } t \text{ to } t + \Delta t\end{bmatrix} = H(t,m,n)\Delta t,$

(A3) $P\begin{bmatrix}\text{more than one casualty during} \\ \text{interval of time from } t \text{ to } t + \Delta t\end{bmatrix} = o\left((\Delta t)^2\right),$

(A4) no more casualties can occur once $m = 0$ or $n = 0$.

The reader should observe that assumptions (A1) through (A3) pertain to the casualty process, while (A4) pertains to the battle-termination process. The significant thing to note is that our stochastic combat model integrates together both an attrition-process model and also a battle-termination-process model.

There are many different other battle-termination models (e.g. see Chapter 3) that could be used in our stochastic combat model. We will only consider one of them here, though: we will assume that the battle terminates when one side's force level reaches a fixed "breakpoint" value (fixed-force-level-breakpoint battle) (see Sections 3.2 and 6.6). It follows that the force level of the other side (i.e. the winner) will always have been above its breakpoint value.

For such a fixed-force-level-breakpoint battle, the above forward KOLMOGOROV equations take a slightly different form on the boundary of the state space, which itself is different. For the deterministic model we assume that $dx/dt = 0$ and $dy/dt = 0$ when $x = x_{BP}$ or $y = y_{BP}$, where x_{BP} denotes X's breakpoint force level and y_{BP} denotes that of Y. The corresponding assumption for the stochastic model is that no more casualties can occur when $m = m_{BP}$ or $n = n_{BP}$. Here m_{BP} denotes X's fixed-force-level-breakpoint for this MARKOV-chain model and similarly for n_{BP}. Making the standard assumptions (A1) through (A3) and

(A4') no more casualties can occur once $m = m_{BP}$ or $n = n_{BP}$,

we may develop the following forward KOLMOGOROV equations for such a battle

for $m = m_0$ and $n = n_0$

$$\frac{dP}{dt}(t,m_0,n_0) = -\{G(t,m_0,n_0) + H(t,m_0,n_0)\} P(t,m_0,n_0) , \tag{4.3.20}$$

for $0 \leq m_{BP} < m < m_0$ and $n = n_0$

$$\frac{dP}{dt}(t,m,n_0) = G(t,m+1,n_0) P(t,m+1,n_0) -\{G(t,m,n_0) + H(t,m,n_0)\} P(t,m,n_0), \tag{4.3.21}$$

for $m = m_0$ and $0 \leq n_{BP} < n < n_0$

$$\frac{dP}{dt}(t,m_0,n) = H(t,m_0,n+1) P(t,m_0,n+1) -\{G(t,m_0,n) + H(t,m_0,n)\} P(t,m_0,n), \tag{4.3.22}$$

for $m_{BP} < m < m_0$ and $n_{BP} < n < n_0$

$$\frac{dP}{dt}(t,m,n) = G(t,m+1,n) P(t,m+1,n) + H(t,m,n+1) P(t,m,n+1) -\{G(t,m,n) + H(t,m,n)\} P(t,m,n), \tag{4.3.23}$$

for $m = m_{BP}$ and $n_{BP} < n \leq n_0$

$$\frac{dP}{dt}(t,m_{BP},n) = G(t,m_{BP}+1,n) P(t,m_{BP}+1,n) , \tag{4.3.24}$$

for $m_{BP} < m \leq m_0$ and $n = n_{BP}$

$$\frac{dP}{dt}(t,m,n_{BP}) = H(t,m,n_{BP}+1)\,P(t,m,n_{BP}+1), \tag{4.3.25}$$

and for $m = m_{BP}$ and $n = n_{BP}$

$$P(t,m_{BP},n_{BP}) = 0 \qquad \text{for all} \quad t \geq 0, \tag{4.3.26}$$

with initial conditions (4.3.4). This system of forward KOLMOGOROV equations (4.3.20) through (4.3.26) with initial conditions (4.3.4) is a stochastic version of the deterministic combat-attrition model (4.2.1) with battle-termination conditions that dx/dt and $dy/dt = 0$ when either $x = x_{BP}$ or $y = y_{BP}$. A fight-to-the-finish is a special case of these equations. Thus, when $m_{BP} = 0$ and $n_{BP} = 0$ in equations (4.3.20) through (4.3.26) this model reduces to equations (4.3.13) through (4.3.19).

4.4. Information to be Obtained from the Model.

The above forward KOLMOGOROV equations (4.3.13) through (4.3.19) with initial conditions (4.3.4) comprise our formulation of a stochastic analogue of the deterministic LANCHESTER-type combat model (4.2.1). The analysis of such a model should be guided by what information one would like to obtain from the model. Conversely, the analytical results that have appeared in the literature are a reflection of such considerations and may therefore be placed in proper perspective by considering the question of what information to extract from the model. Furthermore, such questions are equally valuable for guiding computational work in those cases in which the model is not particularly tractable analytically.

What information should we seek to obtain from a stochastic combat model? Although the specific information to extract from any combat model depends, of course, on the purpose of the OR study using that model, one can anticipate such demands by considering the questions shown in Table 4.I. Analogous questions for a deterministic combat model are given in Table 6.I (see Section 6.3). Basically, we are interested in what will happen in the battle according to the stochastic model and how this compares with that according to the corresponding deterministic model. In fact, because it is relatively so much more difficult (recall that Footnote 6 has told us that there are many more equations for the stochastic model) to extract such information from stochastic combat models, a reasonable analysis strategy appears to be for one to become familiar with the dynamics of the deterministic model and how those of a corresponding stochastic model differ (both in terms of the mean path of battle and also in terms of stochastic variations about this).

Table 4.I. Information to Extract from Stochastic Combat Model.

(Q1) What is the probability that a given side will "win" the engagement? Be annihilated?

(Q2) How does the probability distribution of the number of survivors on each side change during the course of the battle? How do the average force levels change over time in the battle? What is the variability in battle outcomes about these averages?

(Q3) What is the probability distribution of the numbers of final survivors? What are the expected numbers of final survivors?

(Q4) How does a side's probability of winning vary with changes in the initial force ratio?

(Q5) What is the probability distribution of the length of the battle? How long will the battle last on the average?

(Q6) How does the battle's outcome for the stochastic combat model compare to that of a corresponding deterministic model?

In the remainder of this chapter we will consider answering the questions shown in Table 4.I. One important question, however, that we will not consider is question (Q5). The interested reader can find results on the moments of the distribution of the length of battle in SPRINGALL's Ph.D. thesis [77, pp. 50-54].

4.5. Verification that $P(t,m,n)$ Is a Probability Mass Function.

It will be instructive for us to verify that the solution $P(t,m,n)$ to the forward KOLMOGOROV equations is indeed a probability distribution. This development is particularly significant because it will indicate how we may compute the moments of the joint probability distribution of the number of survivors on each side.

If $P(t,m,n)$ is a probability mass function, then we must have

$$\sum_{m=m_{BP}}^{m_0} \sum_{n=n_{BP}}^{n_0} P(t,m,n) = 1 \qquad \text{for all } t \geq 0. \tag{4.5.1}$$

Let us denote $\sum_{m=m_{BP}}^{m_0} \sum_{n=n_{BP}}^{n_0} P(t,m,n)$ as $\Sigma(t)$. It suffices to show that

$$\frac{d\Sigma}{dt}(t) = 0 \qquad \text{for all } t \geq 0, \tag{4..5.2}$$

with

$$\Sigma(0) = 1 . \tag{4.5.3}$$

The latter condition (4.5.3) readily follows from the definition of $\Sigma(t)$ and (4.3.4). Also, from the definition of $\Sigma(t)$ we obtain

$$\frac{d\Sigma}{dt}(t) = \sum_{m=m_{BP}}^{m_0} \left\{ \sum_{n=n_{BP}}^{n_0} \frac{dP}{dt}(t,m,n) \right\} . \tag{4.5.4}$$

Substituting (4.3.13) through (4.3.19) into (4.5.4) and simplifying, we obtain

$$\frac{d\Sigma}{dt}(t) = \sum_{m=m_{BP}}^{m_0-1} \left\{ \sum_{n=n_{BP}+1}^{n_0} G(t,m+1,n)\ P(t,m+1,n) \right\}$$

$$- \sum_{m=m_{BP}+1}^{m_0} \left\{ \sum_{n=n_{BP}+1}^{n_0} G(t,m,n)\ P(t,m,n) \right\}$$

$$+ \sum_{m=m_{BP}+1}^{m_0} \left\{ \sum_{n=n_{BP}}^{n_0-1} H(t,m,n+1)\ P(t,m,n+1) \right\}$$

$$- \sum_{m=m_{BP}+1}^{m_0} \left\{ \sum_{n=n_{BP}+1}^{n_0} H(t,m,n)\ P(t,m,n) \right\} . \qquad (4.5.5)$$

Transforming indices in two of the summations in (4.4.5.), we obtain,

$$\frac{d\Sigma}{dt}(t) = \sum_{j=m_{BP}+1}^{m_0} \left\{ \sum_{n=n_{BP}+1}^{n_0} G(t,j,n)\ P(t,j,n) \right\}$$

$$- \sum_{m=m_{BP}+1}^{m_0} \left\{ \sum_{n=n_{BP}+1}^{n_0} G(t,m,n)\ P(t,m,n) \right\}$$

$$+ \sum_{m=m_{BP}+1}^{m_0} \left\{ \sum_{k=n_{BP}+1}^{n_0} H(t,m,k)\ P(t,m,k) \right\}$$

$$- \sum_{m=m_{BP}+1}^{m_0} \left\{ \sum_{n=n_{BP}+1}^{n_0} H(t,m,n)\ P(t,m,n) \right\} ,$$

whence follows (5.4.2) and hence (4.5.1).

In a similar fashion, one can show that

$$\frac{d}{dt} E[g(M)\ h(N)] =$$

$$= - \sum_{m=m_{BP}+1}^{m_0} \{g(m)-g(m-1)\} \sum_{n=n_{BP}+1}^{n_0} h(n)\ G(t,m,n)\ P(t,m,n)$$

$$- \sum_{m=m_{BP}+1}^{m_0} g(m) \sum_{n=n_{BP}+1}^{n_0} \{h(n)-h(n-1)\}\ H(t,m,n)\ P(t,m,n)\ . \qquad (4.5.6)$$

This result (4.5.6) is significant, since it allows us to readily compute the average force levels and their variabilities for our LANCHESTER-type MARKOV-chain combat model (4.3.20) through (4.3.26) with initial conditions (4.3.4). We observe that (4.5.2) corresponds to the special case of (4.5.6) in which $g(m) = 1$ and $h(n) = 1$.

The development of (4.5.6) is as follows. First we observe that

$$\frac{d}{dt} E[g(M)\ h(N)] = \sum_{m=m_{BP}}^{m_0} g(m) \left\{ \sum_{n=n_{BP}}^{n_0} h(n) \frac{dP}{dt}(t,m,n) \right\} . \qquad (4.5.7)$$

Substituting (4.3.20) through (4.3.26) into (4.5.7) and simplifying, we obtain

$$\frac{d}{dt} E[g(M)\ h(N)]$$

$$= \sum_{m=m_{BP}}^{m_0-1} g(m) \sum_{n=n_{BP}+1}^{n_0} h(n)\ G(t,m+1,n)\ P(t,m+1,n)$$

$$- \sum_{m=m_{BP}+1}^{m_0} g(m) \sum_{n=n_{BP}+1}^{n_0} h(n)\ G(t,m,n)\ P(t,m,n)$$

$$+ \sum_{m=m_{BP}+1}^{m_0} g(m) \sum_{n=n_{BP}}^{n_0-1} h(n)\ H(t,m,n+1)\ P(t,m,n+1)$$

$$- \sum_{m=m_{BP}+1}^{m_0} g(m) \sum_{n=n_{BP}+1}^{n_0} h(n)\ H(t,m,n)\ P(t,m,n). \qquad (4.5.8)$$

Transforming indices in two of the summations in (4.5.8), we obtain

$$\frac{d}{dt} E[g(M)\ h(N)]$$

$$= \sum_{j=m_{BP}+1}^{m_0} g(j-1) \sum_{n=n_{BP}+1}^{n_0} h(n)\ G(t,j,n)\ P(t,j,n)$$

$$- \sum_{m=m_{BP}+1}^{m_0} g(m) \sum_{n=n_{BP}+1}^{n_0} h(n)\ G(t,m,n)\ P(t,m,n)$$

$$+ \sum_{m=m_{BP}+1}^{m_0} g(m) \sum_{k=n_{BP}+1}^{n_0} h(k-1)\ H(t,m,k)\ P(t,m,k)$$

$$- \sum_{m=m_{BP}+1}^{m_0} g(m) \sum_{n=n_{BP}+1}^{n_0} h(n)\ H(t,m,n)\ P(t,m,n),$$

whence follows (4.5.6).

4.6. The Distribution of Times Between Casualties for the General Model.

The distribution of times between casualties is a basic ingredient for much analysis of our MARKOV-chain model, and it therefore seems appropriate for us to develop it for the general model of Section 4.3. We begin by developing the probability that no losses occur during a time-interval of length t. For $m = m_0$ and $n = n_0$, we have from (4.3.9) [equivalently, (4.3.13)]

$$\frac{dP}{dt}(t,m_0,n_0) = -\{G(t,m_0,n_0) + H(t,m_0,n_0)\}\, P(t,m_0,n_0), \tag{4.6.1}$$

with initial condition $P(0,m_0,n_0) = 1$.

The above differential equation (4.6.1) is readily integrated to yield

$$P(t,m_0,n_0) = \exp\{-\int_0^t [G(s,m_0,n_0) + H(s,m_0,n_0)]ds\} . \tag{4.6.2}$$

We finally observe that

$$P[\text{no casualty by time } t] = P(t,m_0,\, n_0) . \tag{4.6.3}$$

Now let T_1 denote the time at which the first casualty occurs (a r.v.). Since the battle begins at $t = 0$, T_1 is also the length of time until the occurrence of the first casualty in the battle. Then

$$P[T_1 > t] = P[\text{no casualty by time } t] ,$$

and the distribution function (d.f.) for the time until the first casualty $F_{T_1}(t) = P[T_1 \leq t]$ is given by

$$P[T_1 \leq t] = 1 - P[T_1 > t] ,$$

or

$$F_{T_1}(t) = 1 - \exp\{-\int_0^t [G(s,m_0,n_0) + H(s,m_0,n_0)]ds\} . \qquad (4.6.4)$$

The average time until the first casualty $\bar{t}_1 = E[T_1] = \int_0^\infty t\, f_{T_1}(t)dt$, where $f_{T_1}(t) = dF_{T_1}/dt$ denotes the probability density function (p.d.f.) for T_1, is given by

$$\bar{t}_1 = \int_0^\infty t\lambda_C(t) \exp\{-\int_0^t \lambda_C(s)ds\}dt , \qquad (4.6.5)$$

where $\lambda_C(t) = G(t,m_0,n_0) + H(t,m_0,n_0)$ denotes the total casualty rate for both sides. When $\lambda_C(t)$ is constant, say $\lambda_C(t) = \lambda$, then $\bar{t}_1 = 1/\lambda$, which is a key relation for modelling LANCHESTER attrition-rate coefficients (see Sections 4.7 and 5.1).

The above considerations are readily generalized to apply to the occurrence of any casualty in such a battle. Accordingly, we let $T_{BC}^{m,n}$ denote the time between the occurrences of two successive casualties, measured from the occurrence of the last casualty which took the system to state (m,n) (a r.v.). Then for $m_{BP} < m \leq m_0$ and $n_{BP} < n \leq n_0$

$$P[T_{BC}^{m,n} \leq t \mid \text{last casualty at } t_0]$$

$$= 1 - \exp\{\int_{t_0}^{t_0+t} [G(s,m,n) + H(s,m,n)]ds\} , \tag{4.6.6}$$

where $t_0 = 0$ when $m = m_0$ and $n = n_0$. Here we must consider this conditional distribution, since the X and Y attrition rates G and H change over time [for fixed (m,n)] so that the d.f. for $T_{BC}^{m,n}$ depends on exactly when the last casualty occurred.

We will also let $T_X^{m,n}$ denote the time until the next X casualty, measured from the occurrence of the last casualty which took the system to state (m,n) (a r.v.), and we will similarly define the time until the next Y casualty $T_Y^{m,n}$. The assumed MARKOV property (i.e. see assumption (A1) above in Section 4.3) implies that the random variables $T_X^{m,n}$ and $T_Y^{m,n}$ are independent. It is easily shown that

$$P[T_X^{m,n} \leq t \mid \text{last casualty at } t_0]$$

$$= 1 - \exp\{- \int_{t_0}^{t_0+t} G(s,m,n)ds\} , \tag{4.6.7}$$

and similarly for $P[T_Y^{m,n} \leq t \mid \text{last casualty at } t_0]$. We also note the following conditional expectation

$$E[T_X^{m,n} \mid \text{last casualty at } t_0]$$

$$= \int_0^\infty tG(t_0 + t,m,n) \exp\{- \int_{t_0}^{t_0+t} G(s,m,n)ds\}dt , \tag{4.6.8}$$

and similarly for $E[T_Y^{m,n} \mid \text{last casualty at } t_0]$.

The random variables $T_X^{m,n}$ and $T_Y^{m,n}$ are particularly important, since we may use our knowledge about them to compute the probability that the next casualty will be, for example, an X casualty when it occurs. Let us now develop this probability, which plays a key role in developing the probability of winning. We first observe that

$$P[X \text{ casualty} | \text{casualty occurs (previous one at } t_0)]$$
$$= P[T_X^{m,n} < T_Y^{m,n} | \text{last casualty at } t_0] \ . \tag{4.6.9}$$

Since the continuous random variables $T_X^{m,n}$ and $T_Y^{m,n}$ are independent with known distribution functions, we know from Appendix B that

$$P[T_X^{m,n} < T_Y^{m,n} | \text{last casualty at } t_0]$$
$$= \int_0^\infty F_{T_X^{m,n}}(s)\ f_{T_Y^{m,n}}(s)\ ds, \tag{4.6.10}$$

where $F_{T_X^{m,n}}(t)$ denotes $P[T_X^{m,n} \leq t | \text{last casualty at } t_0]$ and

$f_{T_Y^{m,n}}(t)$ denotes the p.d.f. for $T_Y^{m,n}$. Thus, we find that

$$P[X \text{ casualty} | \text{casualty occurs (previous one at } t_0)]$$
$$= 1 - \int_0^\infty H(t_0+s,m,n)\ \exp\{-\int_{t_0}^{t_0+s} [G(r,m,n) + H(r,m,n)]dr\}ds \ . \tag{4.6.11}$$

For stationary transition probabilities, i.e. G and H independent of t, the latter formula simplifies considerably.

4.7. The Special Case of Stationary Transition Probabilities.

We will now consider the special case of battles in which the total-force attrition rates depend only on the force levels and not explicitly on time. In other words, we will consider battles with stationary transition probabilities represented by time-independent attrition rates

$$
\begin{aligned}
G(t,m,n) &= A(m,n) = \text{rate of attrition of } X \text{ force}, \\
H(t,m,n) &= B(m,n) = \text{rate of attrition of } Y \text{ force}.
\end{aligned}
\tag{4.7.1}
$$

With a few important exceptions (e.g. some general results in Sections 4.12 and 4.14 below), we will consider only this special case in the remainder of this chapter. It is the only case in which analytical results for the questions posed in Section 4.4 are available (and even then results are fragmentary). In particular, it is essentially the only case for which analytical results for the state probabilities (i.e. distribution of the numbers of survivors) and the probability of winning have been obtained. In Chapter 6 we show how difficult it is to obtain analytical results for the corresponding deterministic LANCHESTER-type equations with time-dependent attrition-rate coefficients. Consequently, since there are many more equations for the stochastic model (recall Footnote 6), the reader should not be surprised that, except for results like (4.6.2), general analytical results do not exist for stochastic battles with time-dependent attrition-rate coefficients.

Let us now write out in full the forward KOLMOGOROV equations for our battle with stationary transition probabilities. As we saw in Section 4.3 above, the exact form of the complete system is influenced by the

battle-termination model. For the sake of concreteness, we will consider a fixed-force-level-breakpoint battle, with (as usual) m_{BP} denoting X's breakpoint and n_{BP} denoting Y's breakpoint. In this case the forward KOLMOGOROV equations (written out in full) take the following form

<u>for $m = m_0$ and $n = n_0$</u>

$$\frac{dP}{dt}(t,m_0,n_0) = -\{A(m_0,n_0) + B(m_0,n_0)\}\ P(t,m_0,n_0), \tag{4.7.2}$$

<u>for $0 \leq m_{BP} < m < m_0$ and $n = n_0$</u>

$$\frac{dP}{dt}(t,m,n_0) =$$
$$= A(m+1,n_0)\ P(t,m+1,n_0) - \{A(m,n_0) + B(m,n_0)\}\ P(t,m,n_0)\ , \tag{4.7.3}$$

<u>for $m = m_0$ and $0 \leq n_{BP} < n < n_0$</u>

$$\frac{dP}{dt}(t,m_0,n)$$
$$= B(m_0,n+1)\ P(t,m_0,n+1) - \{A(m_0,n) + B(m_0,n)\}\ P(t,m_0,n)\ , \tag{4.7.4}$$

<u>for $m_{BP} < m < m_0$ and $n_{BP} < n < n_0$</u>

$$\frac{dP}{dt}(t,m,n) = A(m+1,n)\ P(t,m+1,n) + B(m,n+1)\ P(t,m,n+1)$$
$$- \{A(m,n) + B(m,n)\}\ P(t,m,n)\}, \tag{4.7.5}$$

for $m = m_{BP}$ and $n_{BP} < n \leq n_0$

$$\frac{dP}{dt}(t,m_{BP},n) = A(m_{BP}+1,n)\,P(t,m_{BP}+1,n), \tag{4.7.6}$$

for $m_{BP} < m \leq m_0$ and $n = n_{BP}$

$$\frac{dP}{dt}(t,m,n_{BP}) = B(m,n_{BP}+1)\,P(t,m,n_{BP}+1)\ , \tag{4.7.7}$$

and for $m = m_{BP}$ and $n = n_{BP}$

$$P(t,m_{BP},n_{BP}) = 0\ , \tag{4.7.8}$$

with initial conditions (4.3.4). Since (4.7.8) always holds, we will not write it out explicitly when we consider special cases of (4.7.2) through (4.7.8).

The analytical extraction of information from the above general model with stationary transition probabilities about the dynamics of combat is still too difficult to contemplate.[10] It corresponds to obtaining analytical results from a deterministic model like (5.1.1), and we show elsewhere in this monograph that even this is impossible for such a simpler deterministic model. Returning to the above stochastic model, we note that to obtain the state-probability vector (i.e. probability distributions of the number of survivors on each side, which may be

considered to be a basic ingredient of computing such results) one must solve the system of equations (4.7.2) through[11] (4.7.7) with initial conditions (4.3.4). This may be done by, for example, recursively solving the equations by elementary integration means (see Example 4.7.1 below). Although partial results are readily obtained, a general solution for $P(t,m,n)$ that holds for all values of m and n has only been obtained in a few special cases. Moreover, when m and n are large, such an analytical solution becomes too complicated to be of any direct practical use. Nevertheless, some important partial results are easily obtained for the general model.

For time-independent attrition rates, the results of Section 4.6 simplify considerably. We find that

$$P[\text{no casualty by time } t] = \exp[-\{A(m_0,n_0) + B(m_0,n_0)\}t] \ . \qquad (4.7.9)$$

The times between casualties are exponentially distributed (but state dependent) with

$$P[T_{BC}^{m,n} \leq t] = 1 - \exp[-\{A(m,n) + B(m,n)\}t] \ , \qquad (4.7.10)$$

where $T_{BC}^{m,n}$ denotes the time between the occurrences of two casualties (see Section 4.6 for a precise definition of this and the following random variables $T_X^{m,n}$ and $T_Y^{m,n}$). Considering these random variables, we observe that we no longer have to condition on when the last casualty

has occurred, since the attrition rates A and B do not explicitly depend on time. The expected time between casualties is given by

$$E[T_{BC}^{m,n}] = \frac{1}{A(m,n) + B(m,n)} . \tag{4.7.11}$$

Similarly, the time until the occurrence of the next X casualty $T_X^{m,n}$ is also exponentially distributed (but state dependent) with

$$P[T_X^{m,n} \leq t] = 1 - e^{-A(m,n)t} , \tag{4.7.12}$$

and similarly for $T_Y^{m,n}$. We recall that the random variables $T_X^{m,n}$ and $T_Y^{m,n}$ are independent. The expected value of $T_X^{m,n}$ is given by

$$E[T_X^{m,n}] = \frac{1}{A(m,n)} , \tag{4.7.13}$$

and similarly for $T_Y^{m,n}$. Finally, we find that the probability that the next casualty is, for example, an X casualty, which we know is given by P[X casualty|casualty occurs] = $P[T_X^{m,n} < T_Y^{m,n}]$, reduces to [cf. (4.6.11) above]

$$P[X \text{ casualty} | \text{casualty occurs}] = \frac{A(m,n)}{A(m,n) + B(m,n)} . \tag{4.7.14}$$

Again, the time of occurrence of the last casualty does not influence this probability.

Returning now to the state-probability vector, we note that results (i.e. a general solution for the complete state-probability vector) have appeared in the literature for the following special cases of the

attrition rates A and B (see Section 2.12 for a discussion of the physical circumstances hypothesized to yield such attrition processes and also an explanation of notation):

(a) F|F stochastic LANCHESTER-type attrition process

$$\begin{aligned} A(m,n) &= an , \\ B(m,n) &= bm , \end{aligned} \tag{4.7.15}$$

(b) FT|FT stochastic LANCHESTER-type attriton process

$$\begin{aligned} A(m,n) &= amn , \\ B(m,n) &= bmn , \end{aligned} \tag{4.7.16}$$

(c) (F+T)|(F+T) stochastic LANCHESTER-type attrition process

$$\begin{aligned} A(m,n) &= \beta m + an , \\ B(m,n) &= bm + \alpha n . \end{aligned} \tag{4.7.17}$$

A general expression for $P(t,m,n)$, holding for all values of m and n, has only been obtained for the FT|FT stochastic LANCHESTER-type attriton process (see CLARK [16]) and for the (F+T)|(F+T) process for the special case in which $a + \alpha = b + \beta$ (see ISBELL and MARLOW [40]). Other results (e.g. probability of winning) have appeared for the F|FT process:

(d) F|FT stochastic LANCHESTER-type attrition process

$$\begin{aligned} A(m,n) &= an , \\ B(m,n) &= bmn. \end{aligned} \tag{4.7.18}$$

Example 4.7.1. For the F|F stochastic LANCHESTER-type attrition process with time-independent attrition-rate coefficients, the substitution of (4.7.15) into the forward KOLMOGOROV equations (4.7.2) through (4.7.7) yields

for $m = m_0$ and $n = n_0$

$$\frac{dP}{dt}(t,m_0,n_0) = -(an_0 + bm_0)\, P(t,m_0,n_0), \tag{4.7.19}$$

for $0 \leq m_{BP} < m < m_0$ and $n = n_0$

$$\frac{dP}{dt}(t,m,n_0) = an_0 P(t,m+1,n_0) - (an_0 + bm)\, P(t,m,n_0) \tag{4.7.20}$$

for $m = m_0$ and $0 \leq n_{BP} < n < n_0$

$$\frac{dP}{dt}(t,m_0,n) = bm_0\, P(t,m_0,n+1) - (an + bm_0)\, P(t,m_0,n), \tag{4.7.21}$$

for $m_{BP} < m < m_0$ and $n_{BP} < n < n_0$

$$\frac{dP}{dt}(t,m,n) = anP(t,m+1,n) + bmP(t,m,n+1) - (an + bm)\, P(t,m,n), \tag{4.7.22}$$

for $m = m_{BP}$ and $n_{BP} < n \leq n_0$

$$\frac{dP}{dt}(t, m_{BP}, n) = anP(t, m_{BP}+1, n), \qquad (4.7.23)$$

for $m_{BP} < m \leq m_0$ and $n = n_{BP}$

$$\frac{dP}{dt}(t, m, n_{BP}) = bmP(t, m, n_{BP}+1), \qquad (4.7.24)$$

with initial conditions (4.3.4). Recursively solving the above equations (4.7.19) through (4.7.24) "from the top down," one obtains for

$m_{BP} \leq m \leq m_0$

$$P(t, m, n_0) = \frac{1}{J!}\left\{\left(\frac{an_0}{b}\right)(e^{bt} - 1)\right\}^J \exp[-(bm_0 + an_0)t] , \qquad (4.7.25)$$

where $J = m_0 - m$. Similarly, we find that for $n_{BP} \leq n \leq n_0$

$$P(t, m_0, n) = \frac{1}{K!}\left\{\left(\frac{bm_0}{a}\right)(e^{at} - 1)\right\}^K \exp[-(bm_0 + an_0)t] , \qquad (4.7.26)$$

where $K = n_0 - n$. Further results become fantastically complex and are discussed in Section 4.9 below. Let us now, however, indicate how (4.7.25) may be obtained by recursively solving (4.7.19) and (4.7.20) "from the top down" by elementary means.

Figure 4.5 schematically shows that analytical calculation of $P(t,m,n)$ requires previous determination of $P(t,m+1,n)$ and $P(t,m,n+1)$ [cf. (4.7.2) through (4.7.7) or (4.7.19) through (4.7.24)]. This "from-the-top-down" sequence of integration of the forward KOLMOGOROV equations must be followed, of course, for any battle dynamics [recall (4.3.20) through (4.3.25)]. Returning to the specifics of the $F|F$ stochastic LANCHESTER-type attrition process, we see that equation (4.7.19) with the initial condition $P(0,m_0,n_0) = 1$ is readily integrated to yield

$$P(t,m_0,n_0) = \exp[-(bm_0 + an_0)t] \; . \tag{4.7.27}$$

We observe that (4.7.27) is just a special case of (4.6.2) [see also (4.7.9)]. For $m = m_0-1$ and $n = n_0$, equation (4.7.20) with (4.7.27) substituted into it reads

$$\frac{dp}{dt} + \{b(m_0-1) + an_0\}p = an_0 \exp[-(bm_0 + an_0)t] \tag{4.7.28}$$

with initial condition

$$p(0) = 0 \; ,$$

where for convenience we have denoted $P(t,m_0-1,n_0)$ simply as $p(t)$. Multiplying both sides of (4.7.28) by the integrating factor $\exp\,[b(m_0-1) + an_0]t$, we find that

$$\frac{d}{dt}\,\{p(t)\,\exp([b(m_0-1) + an_0]t)\} = an_0e^{-bt} \; ,$$

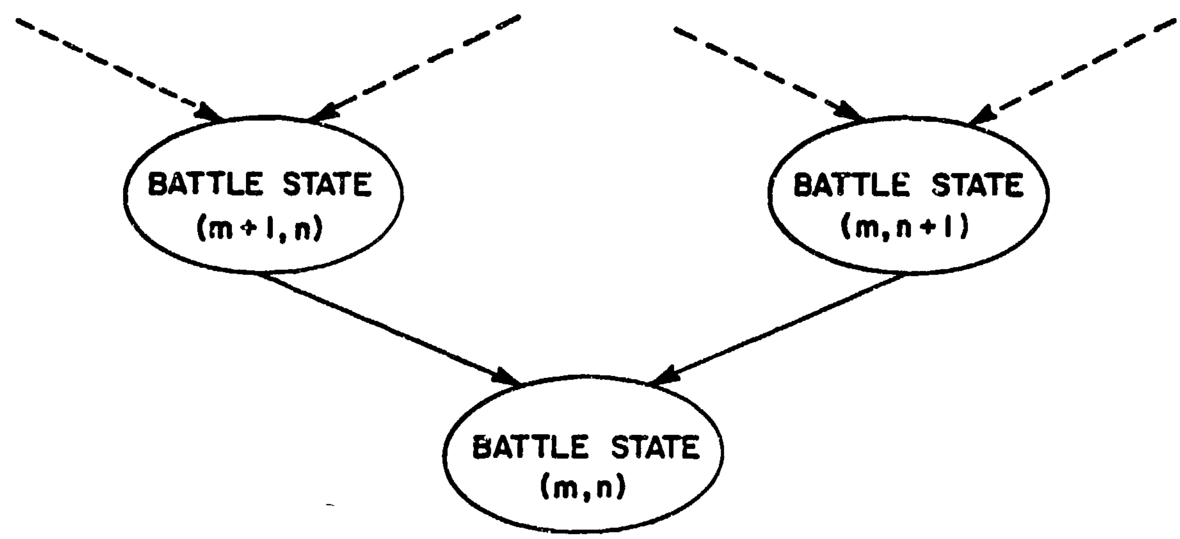


Figure 4.5. Dependence of the calculation of the component P(t,m,n) of the state-probability vector for battle state (m,n) on the previous determination of those for other battle states.

whence

$$P(t,m_0-1,n_0) = \left(\frac{an_0}{b}\right)(e^{bt}-1)\exp[-(bm_0+an_0)t] \ . \tag{4.7.29}$$

Using (4.7.29), we may solve (4.7.20) for $P(t,m_0-2,n_0)$, whence follows (4.7.25) by repeated application of the above procedure.

4.8. An Important Tie-In to the Modelling of LANCHESTER Attrition-Rate Coefficients

When the attrition rates in a battle are independent of time, computation of the average time for the occurrence of, for example, an X casualty suggests that X's loss rate be modelled by the reciprocal of the expected time for the Y force to kill an X target. This simple result is quite important, since it forms the conceptual basis for the analytical modelling of LANCHESTER attriton-rate coefficients (see Chapter 5, especially Section 5.1). Because of its great importance, let us develop (5.1.3), even though it is implicit in previous results of Section 4.7.

For conformance with notation used elsewhere in this book, we will denote $T_X^{m,n}$ simply as T_{XY}. Thus, T_{XY} represents the time (a r.v.) for the entire Y force to kill a single X combatant in our stochastic battle in which casualties occur randomly over time. This time is measured from the occurrence of the last casualty in the battle. Likewise, we will denote $T_Y^{m,n}$ simply as T_{YX}, the time for the entire X force to kill a single Y combatant. For battles with stationary transition probabilities (i.e. the attrition rates depend only on the force levels and not explicitly on time; cf. Section 4.7), the average time for, for example, the Y force to kill an X target takes a particularly enlightning form. First of all, we have shown that T_{XY} is exponentially distributed, with

$$P[T_{XY} \leq t] = 1 - e^{-A(m,n)t} , \qquad (4.8.1)$$

and similarly for T_{YX}. Here the distribution function for T_{XY} does not depend on t_0 [cf. (4.6.7)], since the attrition rate A does not explicitly contain time t. It follows that [cf. (4.7.13)]

$$E[T_{XY}] = \frac{1}{A(m,n)} , \tag{4.8.2}$$

and we similarly find that

$$E[T_{YX}] = \frac{1}{B(m,n)} . \tag{4.8.3}$$

In other words, the reciprocal of the attrition rate is equal to the expected time to kill an enemy combatant. Conversely, the expected time to kill an enemy combatant may be used to predict a numerical value for the corresponding attrition rate, and thus the relations (4.8.2) and (4.8.3) form the conceptual basis for the modelling of LANCHESTER attrition-rate coefficients (see Chapter 5 for further details). They suggest the following "estimators" for the X and Y loss rates

$$\hat{A} = \frac{1}{\bar{t}_{XY}} , \qquad \text{and} \qquad \hat{B} = \frac{1}{\bar{t}_{YX}} , \tag{4.8.4}$$

where $\hat{A}$ denotes an estimate of the X loss rate, $\bar{t}_{XY}$ denotes the average time for the Y force to destroy a single X combatant, and similarly for $\hat{B}$ and $\bar{t}_{YX}$.

In particular, for the F|F LANCHESTER-type attrition process, we have $A(m,n) = an$ and $B(m,n) = bm$, and we may write

$$a = \frac{1}{E[T_{XY}]}, \qquad \text{and} \qquad b = \frac{1}{E[T_{YX}]}, \tag{4.8.5}$$

where T_{XY} now denotes the time for a single Y firer to kill an X target (a r.v.) and similarly for T_{YX}. The above expressions (4.8.5) form the conceptual basis of S. BONDER's modelling of the LANCHESTER attrition-rate coefficients for combat modelled by LANCHESTER-type equations of modern warfare or its heterogeneous-force extension (see Chapter 5 for further details).

4.9. The State Probabilities.

The probability distribution of the joint number of survivors on each side (i.e. the state-probability vector) may be directly computed from the forward KOLMOGOROV equations. From the joint distribution of survivors one can compute many other quantities of interest, e.g. average force levels, distribution of final survivors, etc. (cf. Table 4.I). We will see, however, that such derived quantities can many times be computed more simply by other means, without having to first determine the joint distribution of survivors. Technically speaking, $P(t,m,n)$ is the joint probability distribution of $M(t)$ and $N(t)$. We observe that the set of joint possible battle realizations at any time $t > 0$ may be rather large: there are $(m_0+1-m_{BP}) \times (n_0+1-n_{BP})$ possible battle realizations at any time $t > 0$. Thus, for even rather model numbers of combatants (say m_0 and $n_0 > 10$) the joint distribution of survivors is rather unwieldy. Consequently, even when numerical results are available for the joint distribution of survivors, they are by themselves so uninformative that other measures of battle-state realization are desirable.

Since $P(t,m,n)$ is a joint probability distribution, we know that we must have $0 \leq P(t,m,n) \leq 1$ and

$$\sum_{m=m_{BP}}^{m_0} \sum_{n=n_{BP}}^{n_0} P(t,m,n) = 1 . \qquad (4.9.1)$$

The reader will recall that we have already verified that (4.9.1) indeed holds for the general model with forward KOLMOGOROV equations given by (4.3.13) through (4.3.19).

There are three basic methods[12] for computing the joint probability distribution of the numbers of survivors:

(M1') from an analytical expression,

(M2') by numerical integration of the forward KOLMOGOROV equations,

(M3') by a hybrid analytical-numerical approach (i.e. from an analytical expression with coefficients numerically determined from a system of partial-difference equations[13]).

For convenience, we will refer to these three basic methods simply as follows:

(M1) analytical,

(M2) numerical,

(M3) hybrid.

The analytical method (M1) recursively uses the forward KOLMOGOROV equations to develop an explicit closed-form solution for the state probabilities. Such analytical expressions have been developed in only a few isolated special cases (see below for further details) and then are so complicated that no insights can be directly obtained into the probabilistic dynamics of combat. Furthermore, such analytical expressions are not even apparently

the most computationally efficient (see CLARK [16, p. 115]). The numerical method (M2) uses finite-difference methods (e.g. see HILDEBRAND [37; 38], McCRACKEN and DORN [62], MILNE [63], TODD [80], or Appendix E for further details) to numerically integrate the forward equations. This method always produces numerical results for any given initial numbers of combatants and functional forms of attrition rates but by itself does not directly provide any insights into the dynamics of combat without laboriously grinding out parametric results for judiciously chosen input values (see below for further discussion). Furthermore, such numerical integration is quite computationally inferior to the hybrid method (M3), which in some sense combines the best aspects of the analytical and the numerical methods. The hybrid analytical-numerical method (M3) was apparently first proposed in the Ph.D. thesis of G. M. CLARK [16] and is unfortunately not very widely known. Although somewhat complicated and tedious, it is by far the most computationally efficient approach (see CLARK [16, p. 105]). As with the other methods, it is difficult to obtain insights (without using approximations) into the probabilistic dynamics of combat because of the inherent complexity of results, but the possibilities of this promising approach have not been thoroughly explored.

In the remainder of this section we will focus on reviewing what analytical results have been developed for the joint probability distribution of the numbers of survivors, and we will briefly consider a specific numerical example (with "snapshots" produced by computer graphics of what the joint probability distribution looks like at different points in time over the course of battle). Let us first, however, briefly present CLARK's hybrid method (M3). Connections between results obtainable by this method and

existing analytical results have apparently not been explored at all. In fact, most results in the probabilistic analysis of combat have been more or less ad hoc and isolated. What is needed is a unification and simplification of results, with interrelationships pointed out.

Based on consideration of his specific analytical results for the FT|FT attrition process (see below for specifics), G. M. CLARK [16, p. 106] very insightfully guessed (and then inductively confirmed) that for a fight to the finish modelled by the general stochastic LANCHESTER-type homogeneous-force autonomous combat model [i.e. (4.7.2) through (4.7.8) hold with $m_{BP} = n_{BP} = 0$] the state probabilities are given by

for $0 < m \leq m_0$ and $0 < n \leq n_0$

$$P(t,m,n) = \sum_{j=m}^{m_0} \sum_{k=n}^{n_0} C_{j,k}^{m,n} \exp[-\{A(j,k) + B(j,k)\}t] , \tag{4.9.2}$$

for $0 < m \leq m_0$

$$P(t,m,0) = C_{0,0}^{m,0} + \sum_{j=m}^{m_0} \sum_{k=1}^{n_0} C_{j,k}^{m,0} \exp[-\{A(j,k) + B(j,k)\}t] , \tag{4.9.3}$$

for $0 < n \leq n_0$

$$P(t,0,n) = C_{0,0}^{0,n} + \sum_{j=1}^{m_0} \sum_{k=n}^{n_0} C_{j,k}^{0,n} \exp[-\{A(j,k) + B(j,k)\}t] , \tag{4.9.4}$$

and we recall that $P(t,0,0) \equiv 0$. Using the LAPLACE transform (e.g. see HILDEBRAND [36], PADULO and ARBIB [67], or KLEINROCK [54]), CLARK [16, pp. 109-112] has shown that the constants $C_{j,k}^{m,n}$ are determined by the following system of partial-difference equations

for $0 < m < j \leq m_0$ and $0 < n < k \leq n_0$

$$C_{j,k}^{m,n} = \frac{A(m+1,n)\ C_{j,k}^{m+1,n} + B(m,n+1)\ C_{j,k}^{m,n+1}}{A(m,n) + B(m,n) - A(j,k) - B(j,k)}\ , \tag{4.9.5}$$

for $0 < m < j \leq m_0$ and $0 < n \leq n_0$

$$C_{j,n}^{m,n} = \frac{A(m+1,n)\ C_{j,n}^{m+1,n}}{A(m,n) + B(m,n) - A(j,n) - B(j,n)}\ , \tag{4.9.6}$$

for $0 < m \leq m_0$ and $0 < n < k \leq n_0$

$$C_{m,k}^{m,n} = \frac{B(m,n+1)\ C_{m,k}^{m,n+1}}{A(m,n) + B(m,n) - A(m,k) - B(m,k)}\ , \tag{4.9.7}$$

and for $0 < m \leq m_0$ and $0 < n \leq n_0$ but $(m,n) \neq (m_0,n_0)$

$$C_{m,n}^{m,n} = -\sum_{j=m}^{m_0} \sum_{k=n+1}^{m_0} C_{j,k}^{m,n} - \sum_{j=m+1}^{m_0} C_{j,n}^{m,n}\ , \tag{4.9.8}$$

with $C^{m_0,n_0}_{m_0,n_0} = 1$. Also,

<u>for</u> $0 < m \leq j \leq m_0$ <u>and</u> $1 \leq k \leq n_0$

$$C^{m,0}_{j,k} = - \frac{B(m,1)\, C^{m,1}_{j,k}}{A(j,k) + B(j,k)} \quad , \tag{4.9.9}$$

<u>for</u> $1 \leq m \leq m_0$

$$C^{m,0}_{0,0} = - \sum_{j=m}^{m_0} \sum_{k=1}^{n_0} C^{m,0}_{j,k} \quad , \tag{4.9.10}$$

and similarly for $C^{0,n}_{j,k}$ and $C^{0,n}_{0,0}$. The above expressions (4.9.2) through (4.9.4) are explicit analytical results for the state probabilities, with the constants $C^{m,n}_{j,k}$ determined by (4.9.5) through (4.9.10). CLARK [16] proposed that the constants $C^{m,n}_{j,k}$ be numerically determined by recursive solution of the system of partial-difference equations, and hence we have called this approach the <u>hybrid analytical-numerical method</u>.

It is indeed disappointing that essentially all the analytical results known to this author for the probability distribution $P(t,m,n)$ for such LANCHESTER-type battles are probably best classified as "symbolic," have essentially no computational value, and furthermore provide absolutely no insights into the probabilistic dynamics of combat. Such "symbolic" results are epitomized by the result[14] given by R. H. BROWN [14] for the <u>general stochastic LANCHESTER-type homogeneous- force autonomous</u> (i.e. with time-independent attrition rates) <u>combat model with forward KOLMOGOROV equations</u>

(4.7.2) through (4.7.8). In preparation for stating BROWN's result, we consider a path from the initial state (m_0,n_0) to state (m,n). Such a path in the state space may be described as a sequence of $J = m_0 - m$ zeros and $K = n_0 - n$ ones, where a zero corresponds to a step to the left in the state space (i.e. an X casualty) and a one corresponds to a step down (i.e. a Y casualty) (see Figure 4.6). By considering the binary representation of a positive integer, we (following BROWN [14]) may make correspond to each realization of a battle path an integer k given by

$$k = \delta_{k,1}\,\delta_{k,2} \cdots \delta_{k,J+K}\,, \tag{4.9.11}$$

where (see BROWN [14, pp. 13-14] for further details)

$$\delta_{k,r} = \begin{cases} 1 & \text{if } r\underline{th} \text{ casualty along battle path} \\ & \text{corresponding to } k \text{ is a } Y \text{ combatant,} \\ & \\ 0 & \text{otherwise} \end{cases}$$

Let us denote by $I_{J,K}$ the set of all positive integers whose binary representation contains exactly K ones and no more than J zeroes (again, see BROWN [14, pp. 13-14] for complete details). After r such transitions (provided that $r \leq J + K$), the system will be in state $(m_{k,r}, n_{k,r})$, where

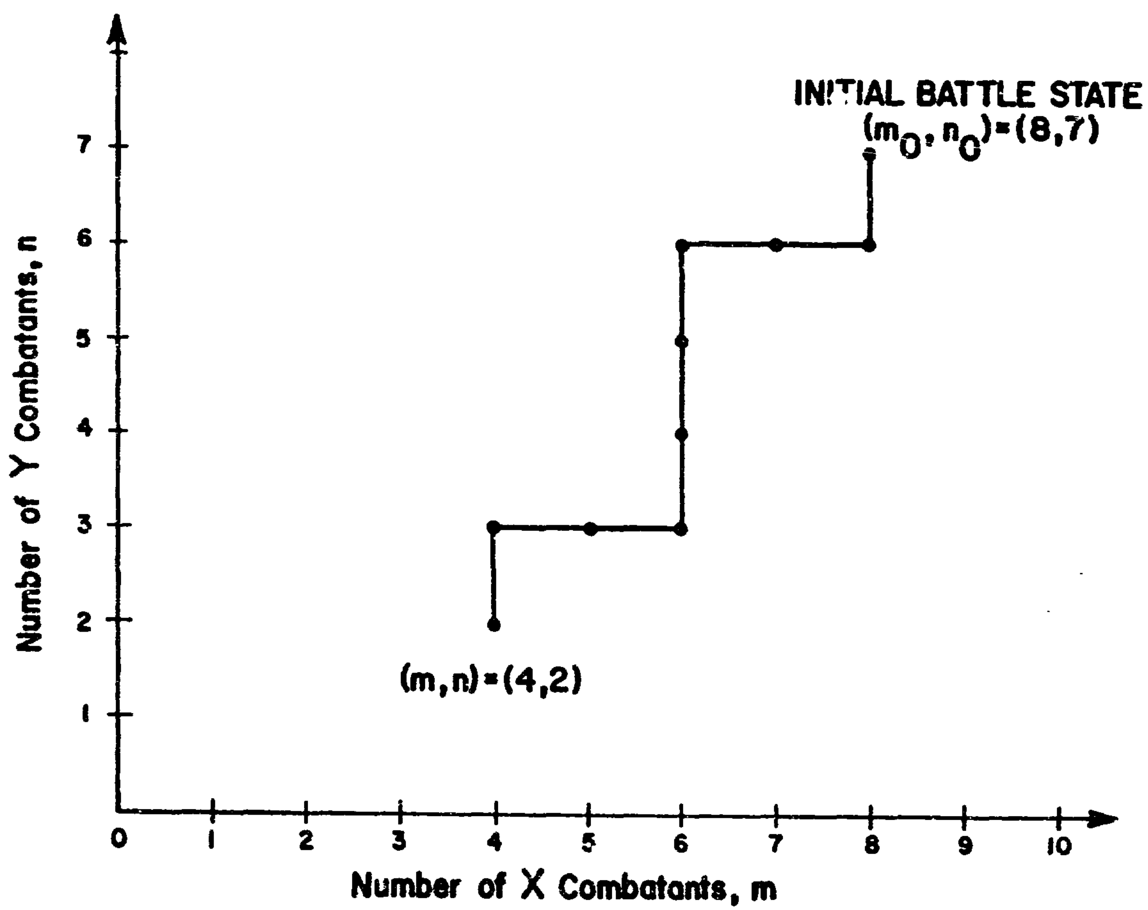


Figure 4.6. Realization of the course of a battle in the state space. Here the occurrence of an X casualty corresponds to a step to the left, while the occurrence of a Y casualty corresponds to a step down.

$$m_{k,r} = m_0 - r + \sum_{j=1}^{r} \delta_{k,j} ,$$

and (4.9.12)

$$n_{k,r} = n_0 - \sum_{j=1}^{r} \delta_{k,j} .$$

Then, BROWN [14, pp. 14-16] shows that

$$P(t,m,n) = \frac{1}{2\pi} \sum_{k\in I_{J,K}} \int_{-\infty}^{\infty} \left\{ \prod_{r=0}^{J+K-1} K_{k,r} \right\} \left\{ \frac{\exp(-iut)-\exp[-t\lambda(m_{k,r},n_{k,r})]}{\lambda(m_{k,r}, n_{k,r}) - iu} \right\} du, \tag{4.9.13}$$

where $i = \sqrt{-1}$ denotes the purely imaginary number of unit magnitude, λ denotes the total casualty rate given by

$$\lambda(m,n) = A(m,n) + B(m,n) , \tag{4.9.14}$$

$$K_{k,r} = \frac{\gamma_{k,r+1}}{1 - iu/\lambda(m_{k,r},n_{k,r})} , \tag{4.9.15}$$

and

$$\gamma_{k,r+1} = \delta_{k,r+1} A(m_{k,r},n_{k,r}) + \{1 - \delta_{k,r+1}\} B(m_{k,r},n_{k,r}) . \tag{4.9.16}$$

BROWN [14, p. 13] points out, though, that unless m is close to m_0 and n is close to n_0, this result is of "little practical interest in the general case."

Although the above expression (4.9.13) is an exact result for the joint probability distribution of the numbers of survivors in the general homogeneous-force LANCHESTER-type battle modelled by a continuous-time MARKOV chain with stationary transition probabilities, the author knows no computational use

(or, for that matter, any use at all) that has ever been made of this imposing formula. Even practical results for special cases of (4.7.2) through (4.7.8) (i.e. for particular functional forms of $A(m,n)$ and $B(m,n)$ in these equations) have been elusive for any and all analytical solution approaches. The difficulty is not in integrating the forward KOLMOGOROV equations for a given initial number of combatants on each side (which can be done by elementary methods in the "top-down" manner discussed in Section 4.7)[15] but in finding a general expression that holds for arbitrary initial numbers of combatants (i.e. for any m_0 and $n_0 > 0$). Let us now examine what analytical results have been obtained by any means for such special cases of the general model (4.7.2) through (4.7.8). We will review essentially all the analytical results known to this author.

The joint probability distribution for the number of survivors on each side has been investigated for the following probabilistic versions of the homogeneous-force battle (4.2.1) with stationary transition probabilities corresponding to the time-independent attrition rates (4.7.1):

(a) <u>FT|FT stochastic LANCHESTER-type attrition process</u>

$$A(m,n) = amn \qquad \text{and} \qquad B(m,n) = bmn\ , \tag{4.9.17}$$

(b) <u>(F+T)|(F+T) stochastic LANCHESTER-type attrition process</u>

$$A(m,n) = an + \beta m \qquad \text{and} \qquad B(m,n) = bm + \beta n, \tag{4.9.18}$$

(c) F|F stochastic LANCHESTER-type attrition process

$$A(m,n) = an \qquad \text{and} \qquad B(m,n) = bm \; . \tag{4.9.19}$$

We will now examine what analytical results for the state-probability vector have been obtained for each of the above three attrition processes.

For the FT|FT stochastic LANCHESTER-type attrition process with attrition rates (4.9.17), G. CLARK [16] has used the LAPLACE transform to find that for $0 < m \le m_0$ and $0 < n \le n_0$

$$P(t,m,n)$$

$$= \frac{a^J b^K}{(a+b)^{J+K}} \sum_{j=m}^{m_0} \sum_{k=n}^{n_0} \frac{(-1)^{j+k-m-n}(m_0)!\,(n_0)!}{m!\,n!\,(k-n)!\,(j-m)!\,(m_0-j)!\,(n_0-k)!}$$

$$\cdot \left\{ \prod_{u=1}^{j-m} \left(\frac{j+n-u}{j+k-u} \right) \prod_{u=1}^{m_0-j} \left(\frac{j+n_0+u}{j+k+u} \right) \right\} \exp[-(a+b)jkt] \, , \tag{4.9.20}$$

where (as above) $J = m_0-m$ and $K = n_0-n$. When $m = 0$, he has also shown [16, pp. 102-103] that for $0 < n \le n_0$

$$P(t,0,n)$$

$$= \frac{a^{m_0} b^K}{(a+b)^{m_0+k}} \sum_{j=1}^{m_0} \sum_{k=n}^{n_0} \frac{(-1)^{j+k-1-n}\,(m_0)!\,(n_0)!}{(n-1)!\,(k-n)!\,j!\,(m_0-j)!\,(n_0-k)!}$$

$$\cdot \left\{ \prod_{u=1}^{j-1} \left(\frac{j+n-u}{j+k-u} \right) \prod_{u=1}^{m_0-j} \left(\frac{j+n_0+u}{j+k+u} \right) \right\} \left\{ \frac{1-\exp[-(a+b)jkt]}{k} \right\} , \tag{4.9.21}$$

and similarly for $P(t,m,0)$. It is clear that no insights into the dynamics of combat are directly obtainable from (4.9.20).

For the (F+T)|(F+T) stochastic LANCHESTER-type attrition process with attrition rates (4.9.18), ISBELL and MARLOW [40] assumed that $a + \alpha = b + \beta$ and found that for $m_{BP} < m \leq m_0$ and $n_{BP} < n \leq n_0$

$$P(t,m,n;0,m_0,n_0)$$

$$= P_{m,n}(m_0,n_0) \binom{m_0+n_0}{J+K} \{\exp[(b+\beta)t]-1\}^{J+K} \exp[-(b+\beta)(m_0+n_0)t] , \qquad (4.9.22)$$

where $J = m_0 - m$ denotes the number of X casualties, similarly for K, and $P_{m,n}(m_0,n_0)$ denotes the probability that the system passes through the transient battle state (m,n) at some time during the battle. In general (i.e for the general attrition process with stationary transition probabilities and no restriction on the attrition-rate coefficients such as $a + \alpha = b + \beta$), this latter probability satisfies the following partial-difference equation for $m_0 \geq m > m_{BP}$ and $n_0 > n > n_{BP}$

$$P_{m,n}(m_0,n_0) = P^X_{NC}(m_0,n_0)\, P_{m,n}(m_0-1,n_0) + P^Y_{NC}(m_0,n_0)\, P_{m,n}(m_0,n_0-1) , \qquad (4.9.23)$$

with boundary conditions

$$P_{m,n}(m_0,n) = \begin{cases} 1 & \text{for } m_0 = m, \\ \prod_{k=1}^{m_0-m} P^X_{NC}(m+k,n) & \text{for } m_0 > m , \end{cases} \qquad (4.9.24)$$

and

$$P_{m,n}(m-1,n_0) = 0 \qquad \text{for } n_0 > n.$$

Here $P_{NC}^{X}(m,n)$ denotes the probability that the next casualty is taken by the X force when the battle state is (m,n), i.e. P[X casualty|casualty occurs] which we have seen is given by (4.7.14), and similarly for $P_{NC}^{Y}(m,n)$. The derivation of the above partial-difference equation for $P_{m,n}(m_0,n_0)$ (4.9.23) is similar to that for $P_{m,n_{BP}}(m_0,n_0)$ (4.10.6) given in Section 4.10 below. It should be noted that, alternatively, we could have taken the partial-difference equation (4.9.23) for $P_{m,n}(m_0,n_0)$ to hold for $m_0 > m$ and $n_0 \geq n$ with the boundary conditions then being

$$P_{m,n}(m,n_0) = \begin{cases} 1 & \text{for } n_0 = n, \\ \prod_{k=1}^{n_0-n} P_{NC}^{Y}(m,n+k) & \text{for } n_0 > n, \end{cases}$$

and (4.9.25)

$$P_{m,n}(m_0,n-1) = 0 \qquad \text{for } m_0 \geq m.$$

For an autonomous (F+T)|(F+T) attrition process, the above equation (4.9.23) becomes [corresponding to the first of the above two equivalent sets of boundary conditions (4.9.24)] for $m_0 \geq m > m_{BP}$ and $m_0 > n > n_{BP}$

$$P_{m,n}(m_0,n_0) = \left\{\frac{\beta m_0 + a n_0}{(b+\beta)m_0 + (a+\alpha)n_0}\right\} P_{m,n}(m_0-1,n_0)$$

$$+ \left\{\frac{b m_0 + \alpha n_0}{(b+\beta)m_0 + (a+\alpha)n_0}\right\} P_{m,n}(m_0,n_0-1), \qquad (4.9.26)$$

with boundary conditions

$$P_{m,n}(m_0,n) = \begin{cases} 1 & \text{for } m_0 = m, \\ \left(1+\frac{\alpha}{a}\right)^{-(m_0-m)} \prod_{k=1}^{m_0-m} \left\{ \dfrac{1+(\beta/an)(k+m)}{1+\left[\dfrac{b+\beta}{n(a+\alpha)}\right](k+m)} \right\} & \text{for } m_0 > m, \end{cases}$$

and (4.9.27)

$$P_{m,n}(m-10,n_0) = 0 \qquad \text{for } n_0 > n.$$

When $a + \alpha = b + \beta$, the above equation (4.9.26) further reduces to (again, for $m_0 \geq m > m_{BP}$ and $n_0 > n > n_{BP}$)

$$P_{m,n}(m_0,n_0) = \left\{ \frac{\beta m_0 + an_0}{(b+\beta)(m_0+n_0)} \right\} P_{m,n}(m_0-1,n_0) + \left\{ \frac{bm_0 + \alpha n_0}{(b+\beta)(m_0+n_0)} \right\} P_{m,n}(m_0,n_0-1) , \tag{4.9.28}$$

with boundary conditions

$$P_{m,n}(m_0,n) = \begin{cases} 1 & \text{for } m_0 = m, \\ \left(1+\frac{\alpha}{a}\right)^{-(m_0-m)} \prod_{k=1}^{m_0-m} \left\{ \dfrac{1+(\beta/an)(k+m)}{1+(1/n)(k+m)} \right\} & \text{for } m_0 > m, \end{cases}$$

and (4.9.29)

$$P_{m,n}(m-1,n_0) = 0 \qquad \text{for } n_0 \geq n .$$

It is worth emphasizing that (4.9.22) holds only when $a + \alpha = b + \beta$, but that (4.9.26) and (4.9.27) hold without any such restriction on the attrition-rate coefficients. Finally, it should be noted that for this special case in which $a + \alpha = b + \beta$ not only can (4.9.22) [after one has analytically solved (4.9.28) with the boundary conditions (4.9.29)] be used to provide an analytical result for $P(t,m,n)$, but it can also be used to provide a hybrid analytical-numerical result (cf. our discussion about the three basic methods for computing the joint probability distribution for the numbers of survivors). In other words, (4.9.22) also provides a basis for a hybrid analytical-numerical approach for computing $P(t,m,n)$ when used in conjunction with a numerical solution (for example, recursively generated with the help of a digital computer) to the partial-difference equation (4.9.28) with the boundary conditions (4.9.29). Thus, we see that there is more than one way to effect a hybrid analytical-numerical solution to the forward KOLMOGOROV equations [cf. CLARK's approach given above and epitomized by (4.9.2)]. It appears that within this context, (4.9.22) provides a more efficient way to compute $P(t,m,n)$ than does (4.9.2), but computational studies are required to confirm this conjecture. In fact, such computational studies are sorely needed in this entire field.

For the F|F stochastic LANCHESTER-type attrition process with attrition rates (4.9.19), the author knows of no general analytical result[16] for the state-probability vector (outside of the partial result given in Section 4.7 above). Before we give an analytical result for the state-probability vector in a special case, it will be convenient for us to first give an important analytical result for $P_{m,n}(m_0,n_0)$, which holds in general for the F|F attrition process and is used to compute $P(t,m,n)$. Accordingly, we observe that GOLDIE [31] has recently shown (using generating functions) that for

the *F|F stochastic LANCHESTER-type attrition process with no restriction on the attrition-rate coefficients* a and b (other than that they be positive), the probability that the system passes through the battle state (m,n) at some time during the battle $P_{m,n}(m_0,n_0)$ is given for $m_0 \geq m > m_{BP}$ and $n_0 \geq n > n_{BP}$ by

$$P_{m,n}(m_0,n_0)$$

$$= \left(\frac{b}{a} m + n\right)\left(\frac{b}{a}\right)^{n_0-n} \sum_{j=m}^{m_0} \frac{(-1)^{m_0-j} \, j^{m_0+n_0-m-n} \, \Gamma(\frac{b}{a} j + n)}{(m_0-j)! \, (j-m)! \, \Gamma(\frac{b}{a} j + n_0 + 1)} . \tag{4.9.30}$$

GOLDIE [31] also gave the following alternate representation for $P_{m,n}(m_0,n_0)$, whose duality with (4.9.30) should be noted.

$$P_{m,n}(m_0,n_0)$$

$$= \left(m + \frac{a}{b} n\right)\left(\frac{a}{b}\right)^{m_0-m} \sum_{k=n}^{n_0} \frac{(-1)^{n_0-k} \, k^{m_0+n_0-m-n} \, \Gamma(m + \frac{a}{b} k)}{(n_0-k)! \, (k-n)! \, \Gamma(m_0 + \frac{a}{b} k + 1)} . \tag{4.9.31}$$

These results may also be obtained by R. H. BROWN's separation-of-variable's method (*see* Appendix C), but the easiest way to obtain them is to use the expression (4.10.2) for $P_{m,n_{BP}}(m_0,n_0)$, the probability that X wins a fixed-force-level-breakpoint battle with m final survivors, and the following recursion (first apparently formally observed by GOLDIE [31])

$$P_{m,n_{BP}}(m_0,n_0) = P_{m,n_{BP}+1}(m_0,n_0) \, P^Y_{NC}(m,n_{BP}+1), \tag{4.9.32}$$

which holds for the general LANCHESTER-type stochastic attrition process with time-independent attrition-rate coefficients and is readily developed by elementary probability arguments. To see how (for example) (4.9.30) may be developed by using (4.9.32), it is convenient to denote the probability $P_{m,n}(m_0,n_0)$ of passing through the transient state (m,n) as $P^T_{m,n}(m_0,n_0)$. Similarly, we will denote the probability $P_{m,n_{BP}}(m_0,n_0)$ of reaching the absorbing state (m,n_{BP}) as $P^A_{m,n_{BP}}(m_0,n_0)$. Using this notation, we may write (4.9.32) more explicitly for $m_0 \geq m > m_{BP}$ and $n_0 \geq n > n_{BP}$ as

$$P^A_{m,n_{BP}} = P^T_{m,n_{BP}+1}(m_0,n_0)\ P^Y_{NC}(m,n_{BP}+1)\ , \tag{4.9.33}$$

whence follows

$$P^T_{m,n}(m_0,n_0) = \frac{P^A_{m,n-1}(m_0,n_0)}{P^Y_{NC}(m,n)} \quad . \tag{4.9.34}$$

The desired result (4.9.30) for $P_{m,n}(m_0,n_0) = P^T_{m,n}(m_0,n_0)$ follows from substituting the expression for $P^A_{m,n-1}(m_0,n_0)$ obtained from (4.10.21) and $P^Y_{NC}(m,n) = \{(b/a)m\}/\{(b/a)m + n\}$ into equation (4.9.34) above. We are now ready to give a result for $P(t,m,n)$ for the F|F stochastic LANCHESTER-type attrition process for the special case of equal attrition-rate coefficients. In this special case in which $a = b$ we may invoke (4.9.22) by setting $\alpha = \beta = 0$, and consequently we find that for $m_0 \geq m > m_{BP}$ and $n_0 \geq n > n_{BP}$

$$P(t,m,n) = P_{m,n}(m_0,n_0)\begin{pmatrix} m_0+n_0 \\ J+K \end{pmatrix} (e^{bt}-1)^{J+K} \exp[-b(m_0+n_0)t], \tag{4.9.35}$$

where (as above) $J = m_0-m$ and $K = n_0-n$. By setting $a = b$ in (4.9.30), we find that $P_{m,n}(m_0,n_0)$ in (4.9.35) is given by

$$P_{m,n}(m_0,n_0) = (m+n) \sum_{j=m}^{m_0} \frac{(-1)^{m_0-j} \, j^{m_0+n_0-m-n} \, (n+j-1)!}{(m_0-j)! \, (j-m)! \, (n_0+j)!} \quad . \tag{4.9.36}$$

Furthermore, F. C. BROOKS [13] has made the very insightful observation that for this special case in which $a = b$ one can very easily obtain simple analytical results for the total number of casualties on both sides. Thus, if the state space is appropriately defined, some very useful information is readily obtained for these stochastic battles. The reader should bear in mind, though, that the basic untractability of quantities like the state-probability vector remain unchanged by such transformations. Thus, BROOKS [13, p. 9] considered the probability that a total of $L = J + K = (m_0-m) + (n_0-n)$ casualties have occurred on <u>both</u> sides by time t, $P_L(t)$, and found that (still for the special case in which $a = b$) for a fight to the finish in which $L < m_0, n_0$

$$P_L(t) = \binom{m_0+n_0}{L} (e^{bt}-1)^L \exp[-b(m_0 + n_0)t] \quad . \tag{4.9.37}$$

It will be instructive for us to show how BROOKS's [13] result (4.9.37) may be obtained from (4.9.27), which is a special case of ISBELL and MARLOW's [40] more general result (4.9.22). To this end, we observe that

$$P_L(t) = \sum_{j=0}^{L} P(t, m_0 - L + j, n_0 - j) , \tag{4.9.38}$$

and hence (4.9.35) yields

$$P_L(t) = S_L(m_0, n_0) \binom{m_0 + n_0}{L} (e^{bt} - 1)^L \exp[-b(m_0 + n_0)t] , \tag{4.9.39}$$

where $L = J + K$ and

$$S_L(m_0, n_0) = \sum_{j=0}^{L} P_{m_0 - L + j, n_0 - j}(m_0, n_0) , \tag{4.9.40}$$

whence (4.9.37) follows from (4.9.35) provided that $S_L(m_0,n_0) = 1.0$. We will now give a probabilistic argument that $S_L(m_0,n_0) = 1.0$, but a direct verification of this fact through use of the above result (4.9.30) for $P_{m,n}(m_0,n_0)$ has so far proven to be elusive. We first observe that

$$S_L(m_0, n_0) = P \begin{bmatrix} \text{a total of } L \text{ casualties on both} \\ \text{sides have occurred at some time} \\ \text{during the course of the battle} \end{bmatrix} . \tag{4.9.41}$$

Assuming that the battle's termination condition involves more than a total of L casualties on both sides, then it is clear that $S_L(m_0,n_0) = 1.0$.

Considering the above results, we begin to gain some appreciation for the great increase in difficulty in analytically extracting information (cf. Table 4.I again) from a simple homogeneous-force combat model by the inclusion of randomness in the attrition process. Especially because of

the combinatorial aspects involved, a modern large-scale, high-speed digital computer must be used to generate numerical (as opposed to analytical) results and can always readily generate such numerical results for a particular set of input values for the battle parameters. Although such particular numerical examples can always be more or less readily generated by a modern digital computer, general insights into the dynamics of combat are again quite difficult to develop and can only be obtained by laboriously grinding out numerical results for given ranges of input values for the battle parameters (see Appendix E for a further discussion of such numerical methods). Nevertheless, at this juncture consideration of a specific computer-generated numerical example should at least provide the reader with some better understanding about the basic nature of probabilistic LANCHESTER-type combat dynamics as portrayed by the joint probability distribution for the numbers of survivors $M(t)$ and $N(t)$, i.e. $P(t,m,n)$.

It is indeed surprising that more use has not been made of the modern large-scale, high-speed digitial computer and associated computer graphics to at least computationally investigate stochastic LANCHESTER-type force-on-force attrition models. Let us now consider such a computer-generated numerical example for the joint probability distribution of $M(t)$ and $N(t)$ for the F|F stochastic LANCHESTER-type attrition process (4.9.10). Numerical results are depicted (for the battle-input data shown in Table 4.II) at five different points in time for this fight to the finish in Figures 4.7 through 4.11. In the corresponding deterministic battle, the X force is annihilated at $t_a^{DX} = 155.81$ minutes; and these plots (i.e. Figures 4.7 through 4.11) correspond to $t = 0.025\ t_a^{DX}$, $0.25\ t_a^{DX}$, $0.50\ t_a^{DX}$, $0.75\ t_a^{DX}$, and $1.0\ t_a^{DX}$, respectively.

TABLE 4.II. Particulars for the Numerical Example for the Evolution of the Joint Probability Distribution for $M(t)$ and $N(t)$ for the $F|F$ Stochastic LANCHESTER-Type Attrition Process (4.9.10) for a Fight to the Finish.

1. <u>Basic Input Data</u>

   $a = 0.008$ X casualties/minute/Y firer

   $b = 0.004$ Y casualties/minute/X firer

   $m_0 = 40$, $n_0 = 40$

   $m_{BP} = 0$, $n_{BP} = 0$

2. <u>Computed Quantities for Corresponding Deterministic Battle</u>

   $t_a^{DX} = 155.81$ minutes [from equation (2.2.20)]

   with $x_f = 0.00$ and $y_f = 28.28$

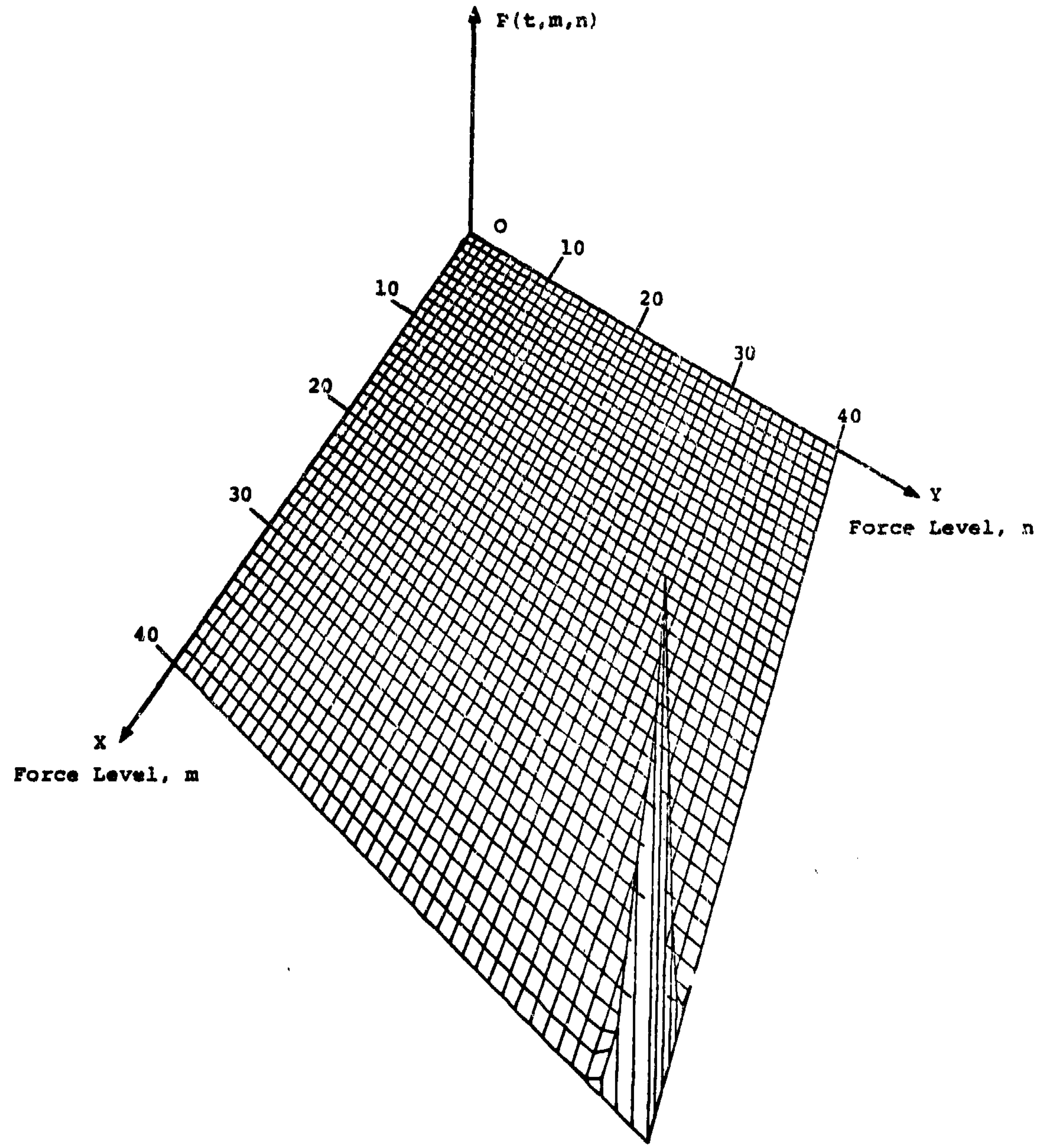


Figure 4.7. Joint probability distribution for M(t) and N(t) for the F|F stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.II at $t = 0.025\ t_a^{DX}$.

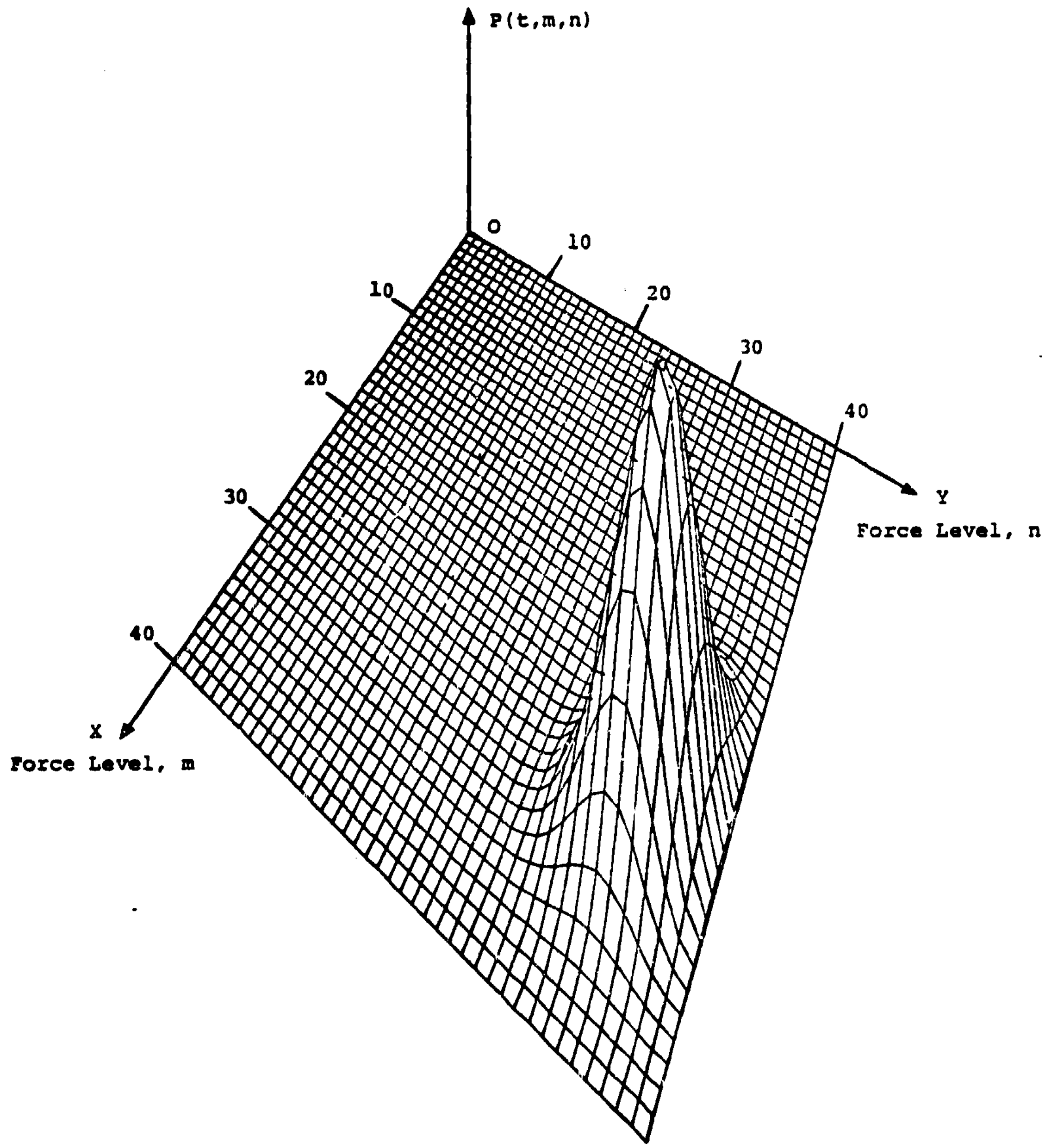


Figure 4.8. Joint probability distribution for M(t) and N(t) for the F|F stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.II at $t = 0.25\ t_a^{DX}$.

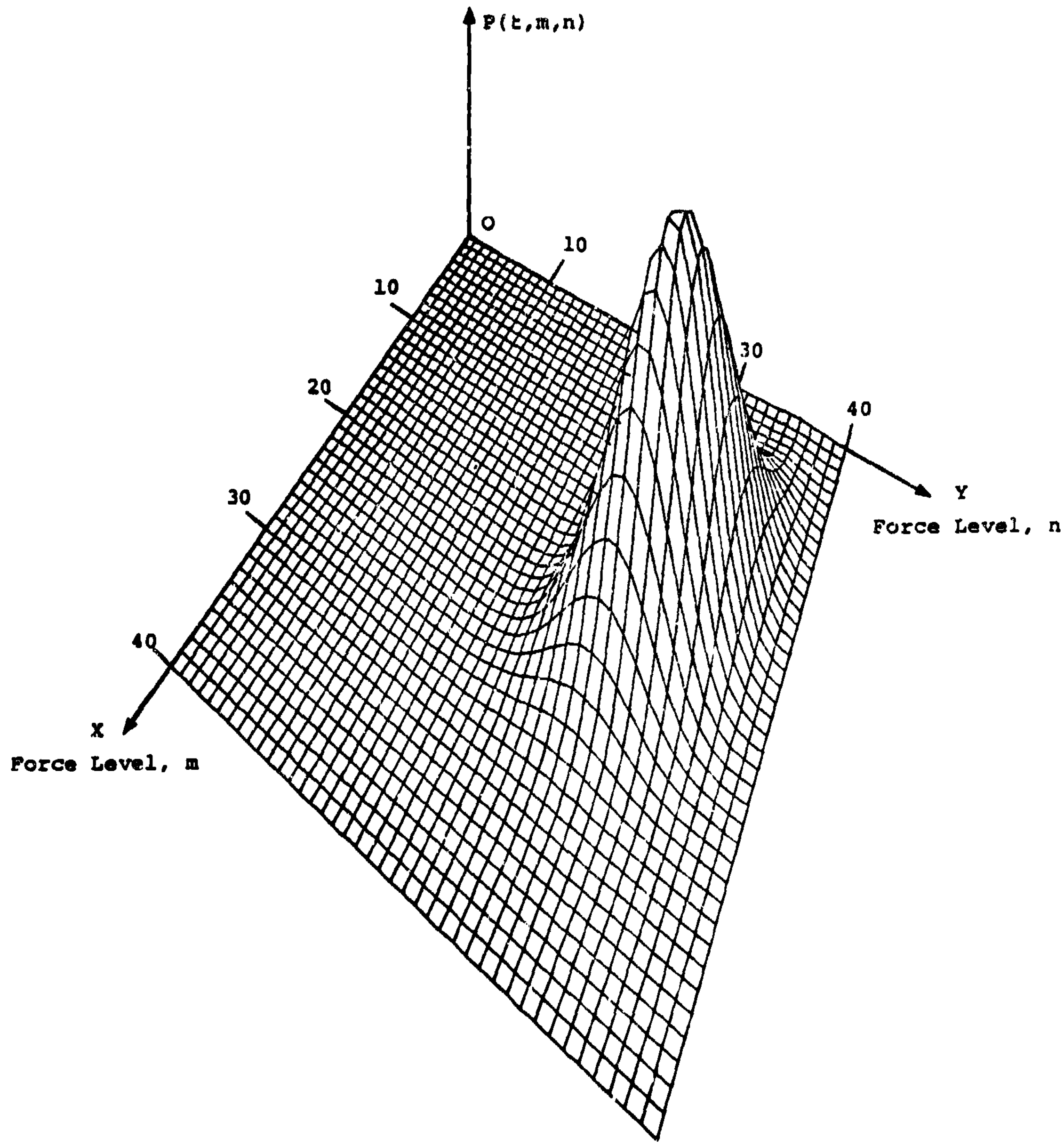


Figure 4.9. Joint probability distribution for M(t) and N(t) for the F|F stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.II at $t = 0.50\ t_a^{DX}$.

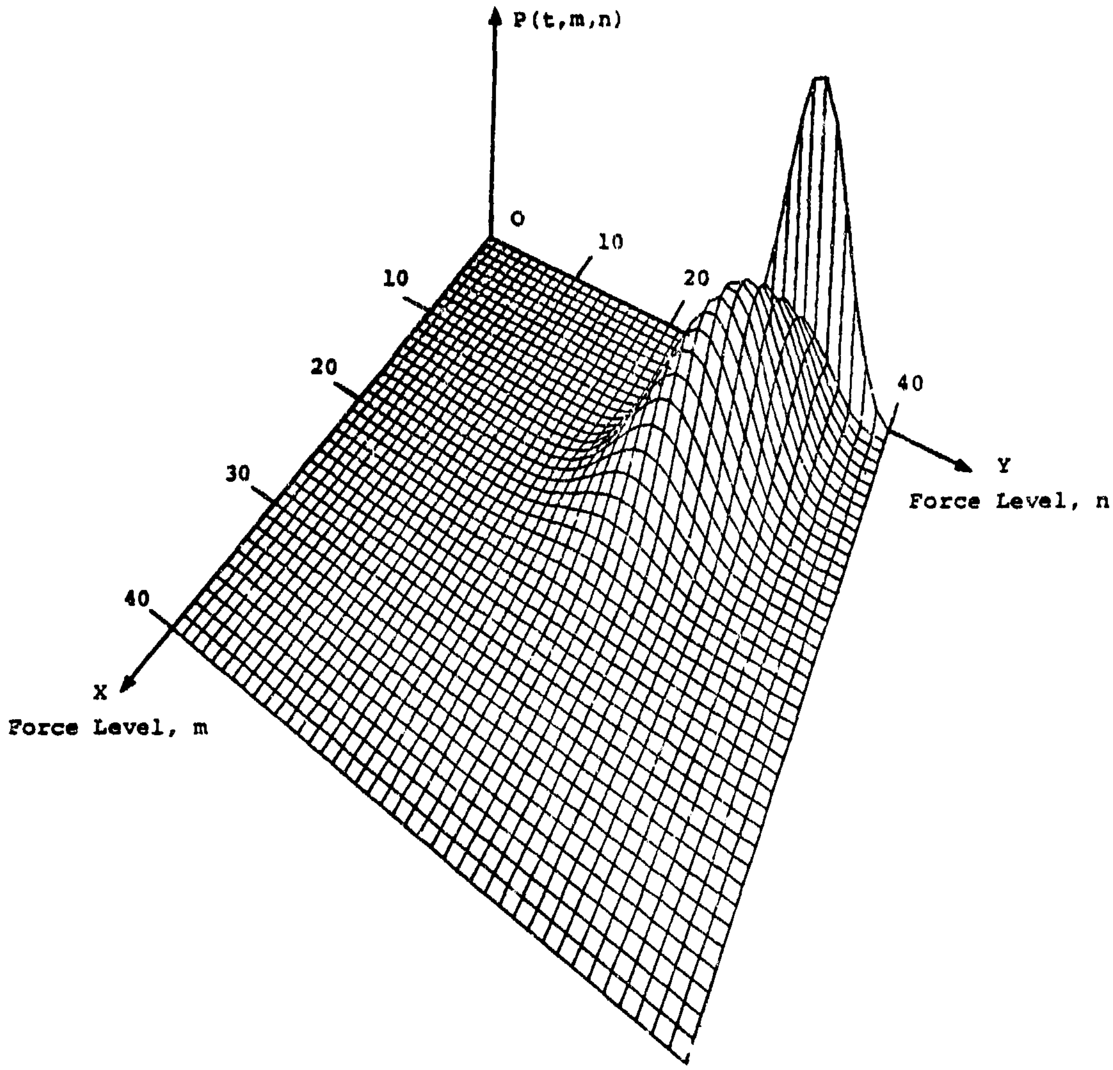


Figure 4.10. Joint probability distribution for M(t) and N(t) for the F|F stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.II at $t = 0.75\ t_a^{DX}$.

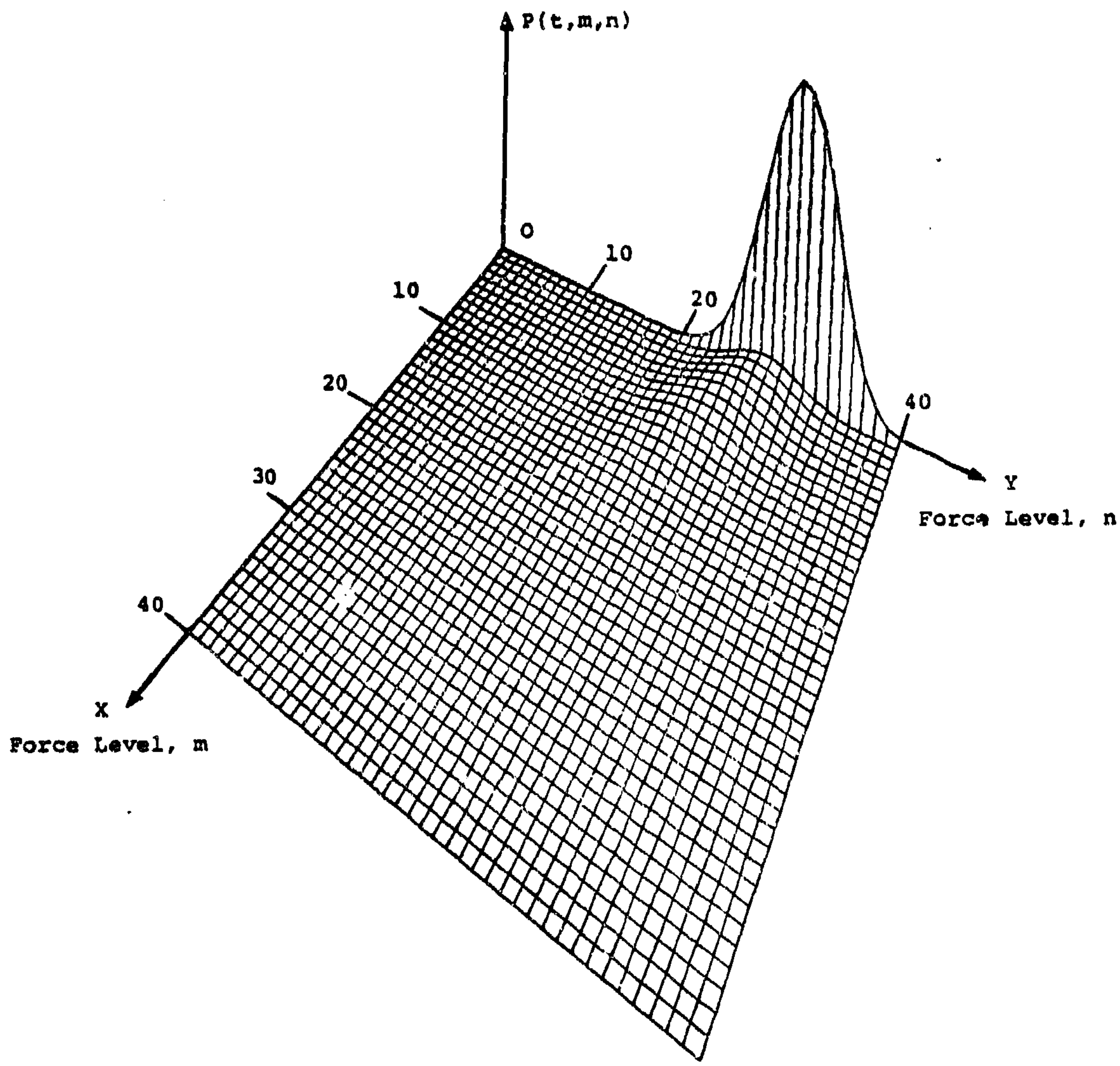


Figure 4.11. Joint probability distribution for M(t) and N(t) for the F|Y stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.II at $t = 1.0\ t_a^{DX}$.

In other words, the time for each of the joint probability distributions shown in Figures 4.7 through 4.11 is expressed in units of t_a^{DX}, how long it takes for the X force to be annihilated in the corresponding battle represented by the deterministic model.

The Figures 4.7 through 4.11 may be thought of as "snapshots" of the joint probability for survivors in this battle taken at a sequence of increasing times. From sequentally looking at these figures, the reader can see how the probability distribution evolves over time. At $t = 0$ all probability is located at the single point $(m_0, n_0) = (40, 40)$ in the state space. By $t = 0.025\ t_a^{DX}$ this "spike" originally at (m_0, n_0) has evolved at a "needle" near (m_0, n_0) (see Figure 4.7). As time goes on, this "needle" of probability mass becomes more and more blunted and moves towards the n-axis (i.e. $m = 0$). The blunting of the probability mass corresponds to diffusion of probability from the mode (i.e. the "high point") of the joint distribution, while the movement of the probability mass in the state space corresponds to convective transport of probability towards the end-of-battle condition in the state space[17] (i.e. annihilation of one side or the other) (see Figures 4.8 through 4.11). Since $(m, 0)$ and $(0, n)$ are absorbing states in this fight to the finish, probability "sticks" to the boundary of the state space as some of the probability mass reaches the boundary (see Figures 4.10 and 4.11). As time increases without bound, all the probability mass becomes absorbed on either the m-axis or the n-axis, and this situation corresponds to the mathematical fact that $\lim_{t \to +\infty} P(t,m,n) = 0$ when both m and $n > 0$. The total amount of probability mass ultimately accumulated on the n-axis is simply the probability that Y wins[18], i.e. $P[Y \text{ wins}] = \sum_{n=1}^{n_0} \lim_{t \to +\infty} P(t,0,n)$, and similarly for $P[X \text{ wins}]$. For the example

at hand, the reader can see from Figure 4.11 that P[Y wins] is rather large and corresponds to a "decisive" win by Y in the deterministic battle. If the opposing forces were closer to "parity" (i.e. the initial force levels were such that in the corresponding deterministic batter the opposing forces would be closer to "parity"), more probability would be absorbed on the n-axis, corresponding to a larger value for P[X wins].

The effect of using deterministic force-level breakpoints in our combat model (cf. Sections 3.2 and 3.4 above) is simply to reduce the state space, with the probability mass moving over time in the same general qualitative manner as in the previous example. To see this in a specific numerical example, let us modify the previous example by changing each side's force-level breakpoint from 0 to 8 (see Table 4.III). The evolution of the joint probability distribution for this fixed-force-level-breakpoint battle for $t = 0.025\ t_W^{DY}$, $0.25\ t_W^{DY}$, $0.50\ t_W^{DY}$, $0.75\ t_W^{DY}$, and $1.0\ t_W^{DY}$ is shown in Figures 4.12 through 4.16 and closely resembles that of the previous example, except that the state space is reduced to (m,n) with $m = 8, 9, \ldots, 40$ and $n = 8, 9, \ldots, 40$. Here t_W^{DY} denotes the duration of the corresponding deterministic battle ("the time for Y to win the deterministic battle") and has been computed according to the result given in Table 2.X for the data shown in Table 4.III (see CRAIG [19] for further computational results).

TABLE 4.III. Particulars for the Second Numerical Example for the Evolution of the Joint Probability Distribution for $M(t)$ and $N(t)$ for the F|F Stochastic LANCHESTER-Type Attrition Process (4.9.10) for a Fixed-Force-Level-Breakpoint Battle.

1. Basic Input Data

$a = 0.008$ X casualties/minute/Y firer

$b = 0.004$ Y casualties/minute/X firer

$m_0 = 40$, $n_0 = 40$

$m_{BP} = 8$, $n_{BP} = 8$

2. Computed Quantities for Corresponding Deterministic Battle

$t_W^{DY} = 120.68$ minutes

with $x_f = 8.00$ and $y_f = 28.84$

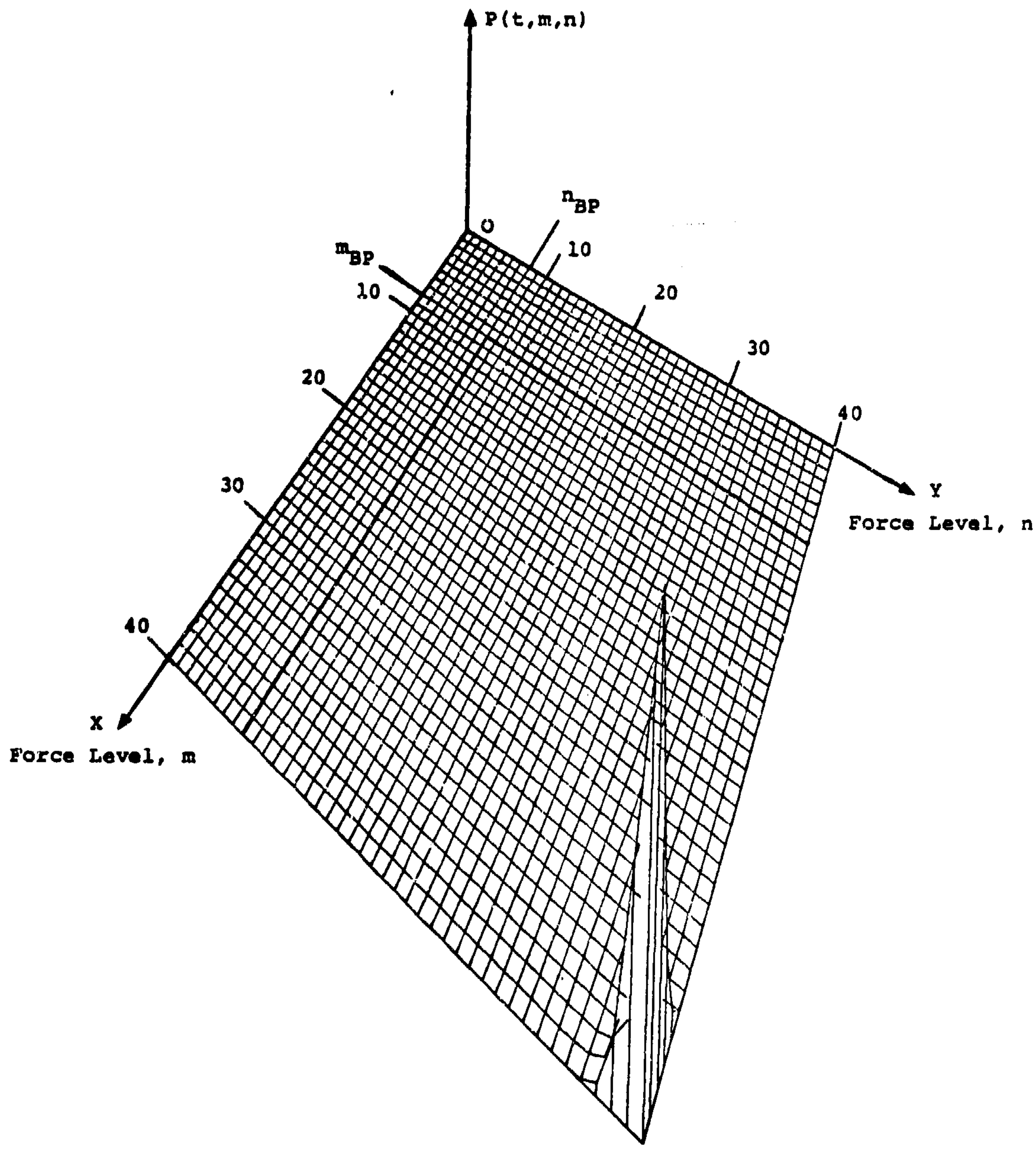


Figure 4.12. Joint probability distribution for $M(t)$ and $N(t)$ for the $F|F$ stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.III at $t = 0.025\ t_W^{DY}$.

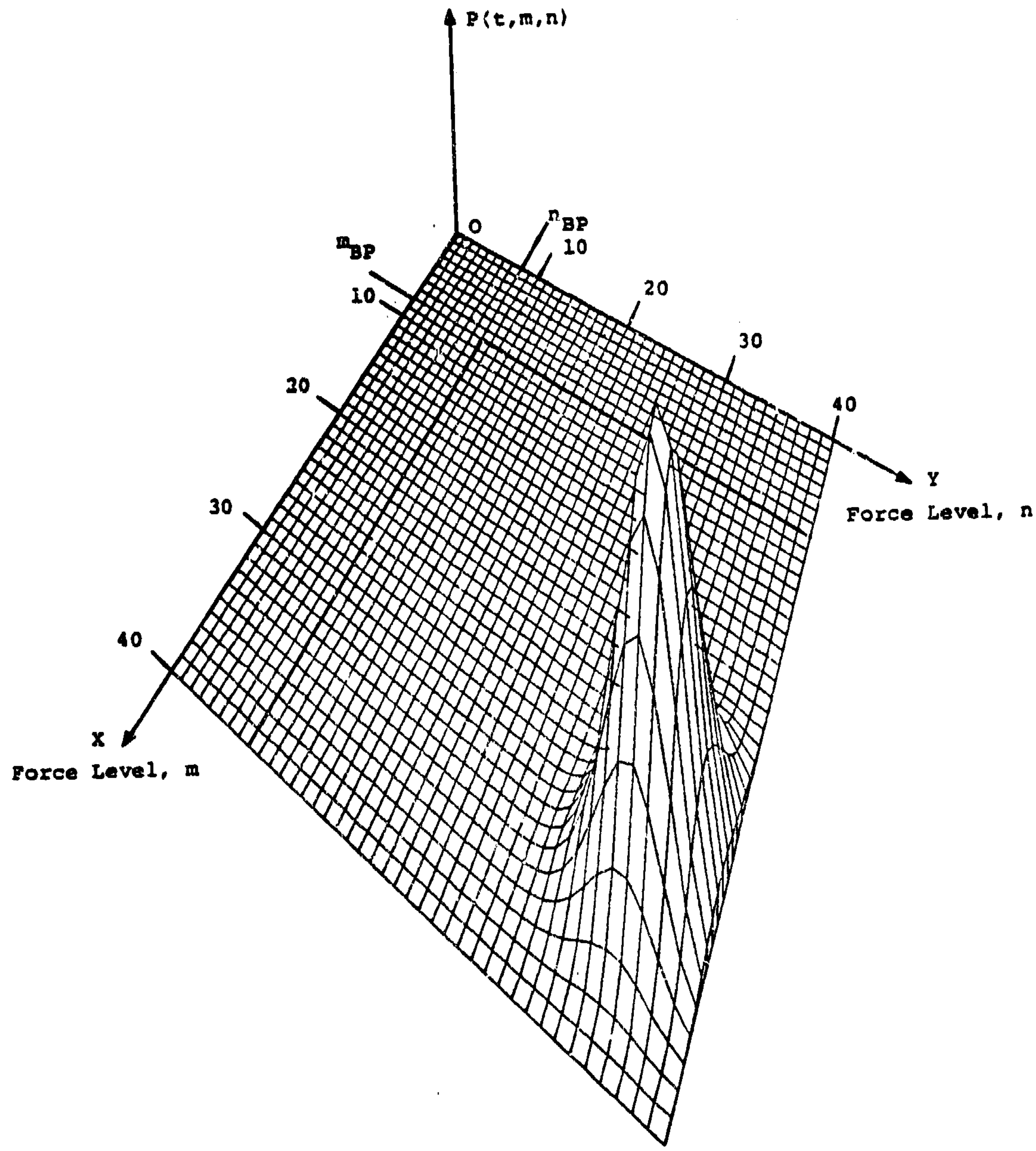


Figure 4.13. Joint probability distribution for $M(t)$ and $N(t)$ for the $F|F$ stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.III at $t = 0.25\ t_W^{DY}$.

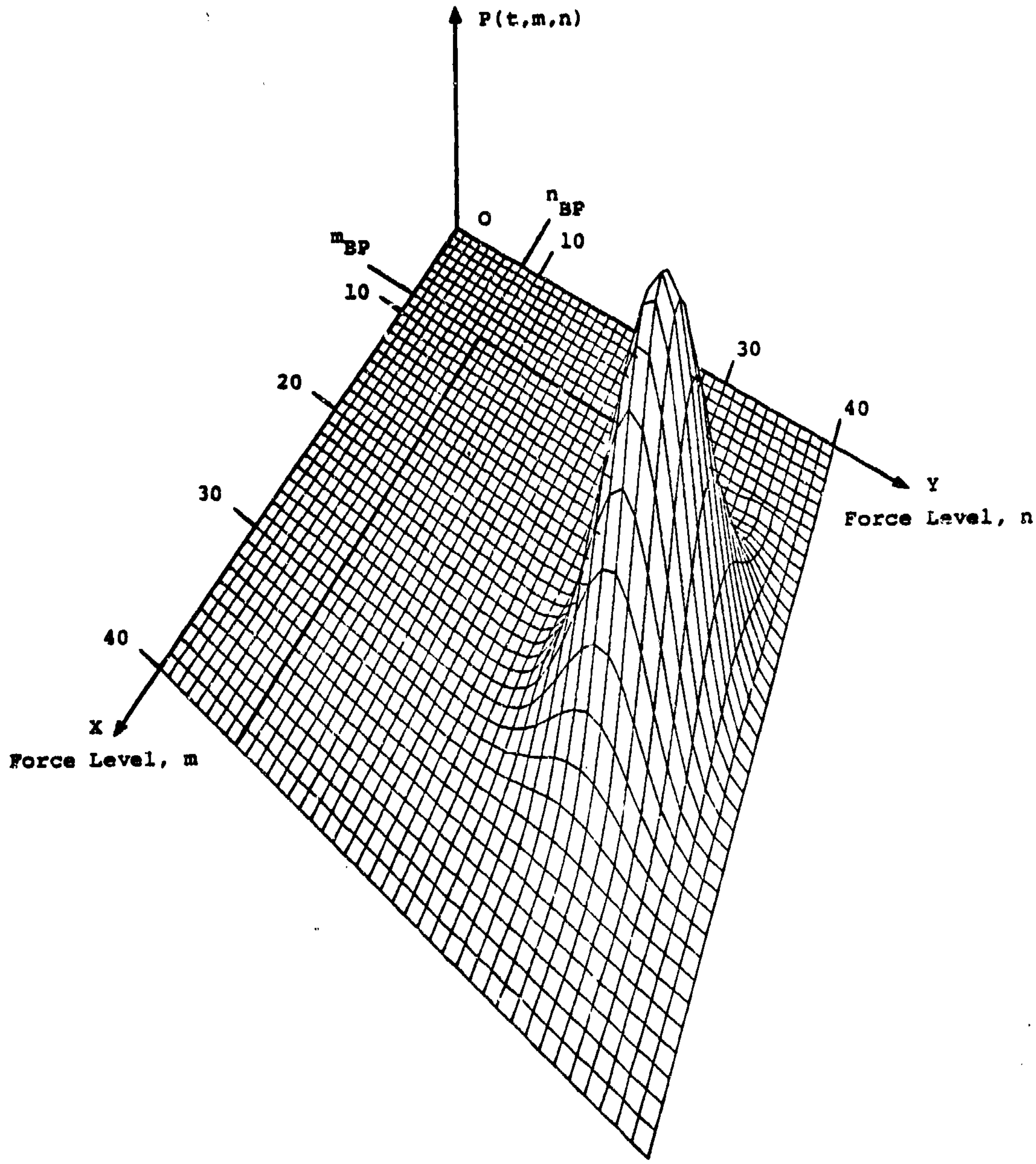


Figure 4.14. Joint probability distribution for $M(t)$ and $N(t)$ for the $F|F$ stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.III at $t = 0.50\, t_W^{DY}$.

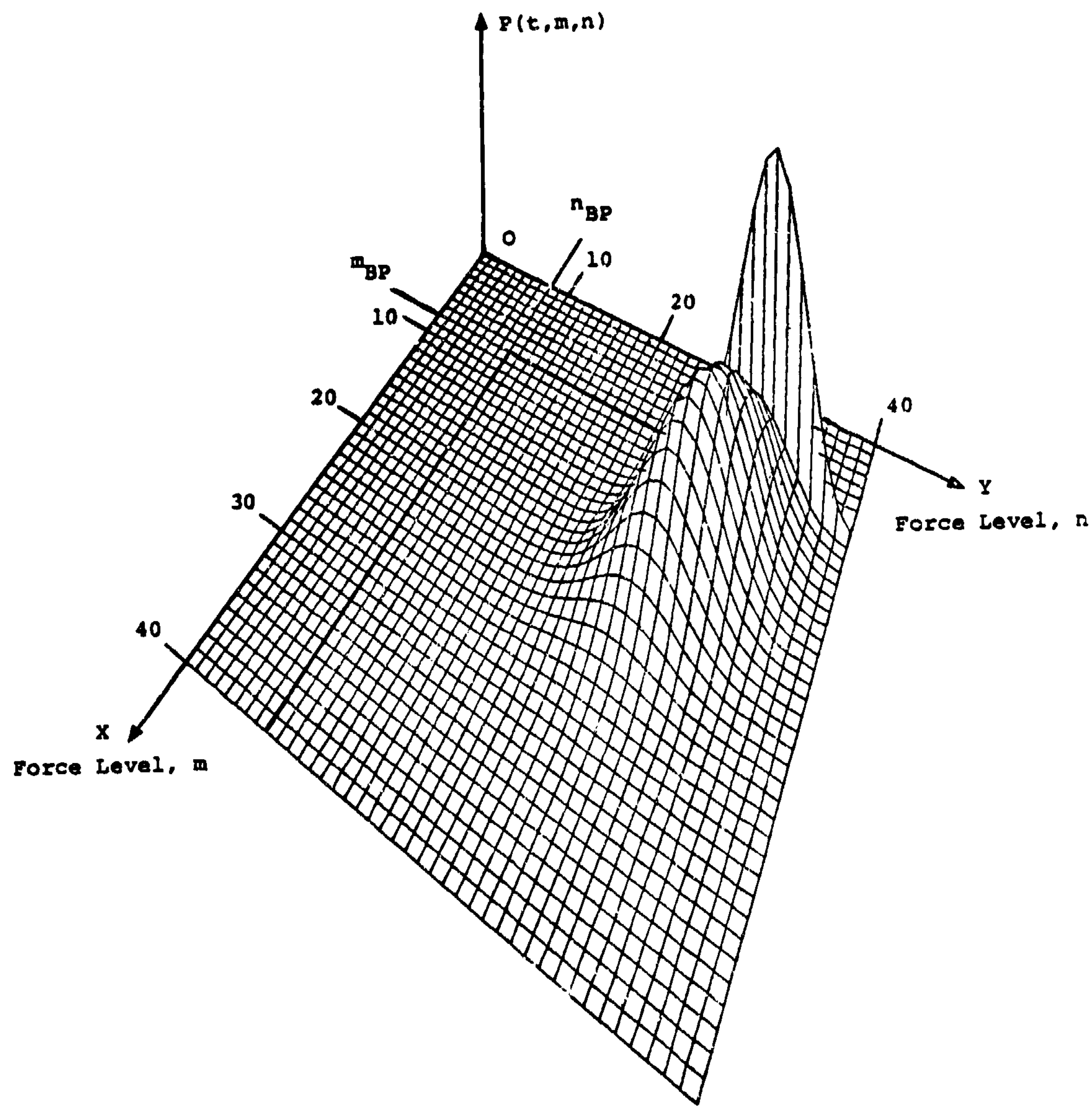


Figure 4.15. Joint probability distribution for $M(t)$ and $N(t)$ for the $F|F$ stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.III at $t = 0.75\ t_W^{DY}$.

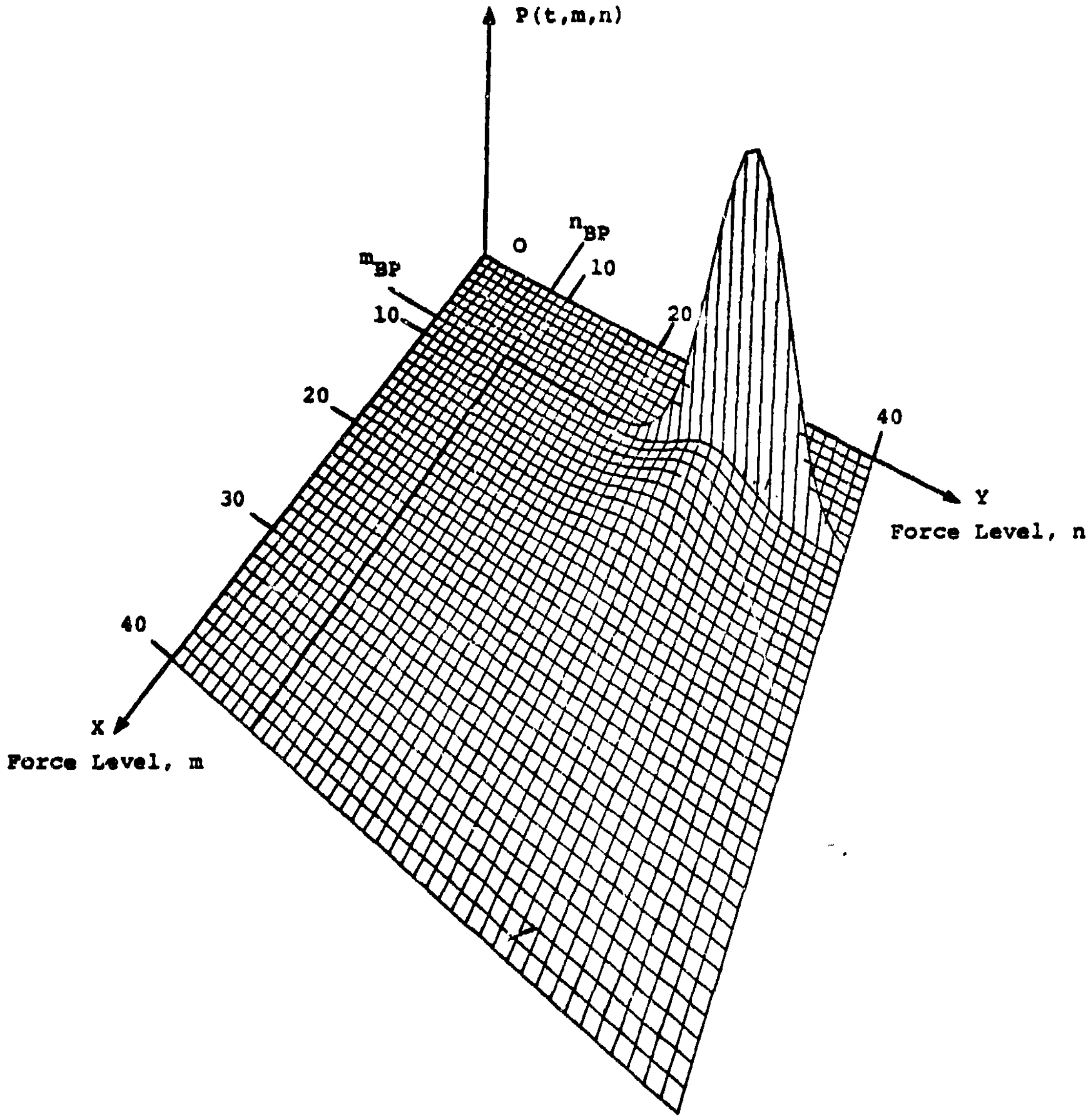


Figure 4.16. Joint probability distribution for $M(t)$ and $N(t)$ for the $F|F$ stochastic LANCHESTER-type attrition process (4.9.10) for the input data given in Table 4.III at $t = 1.0\ t_W^{DY}$.

4.10. The Probability of Winning.

For such a stochastic combat model, other quantities of interest to the combat analyst (recall Table 4.I) are (a) the probability of winning, (b) the distribution of the winner's final survivors at the end of battle, and (c) the expected numbers of final survivors. In this section we will give the fundamental equations that yield the first two important quantities, and then we will give analytical results for all these quantities for the basic homogeneous-force LANCHESTER-type models for the FT|FT, F|F, and F|FT attrition processes. As we have stressed many times above (especially for deterministic attrition processes), such results are heavily dependent on the model taken for battle termination. Moreover, all the results given in this section are for fixed-force-level-breakpoint battles[19] (a special case of which is the fight to the finish).

Thus, in this section we will develop the fundamental partial-difference equations for determining

(I) the probability of winning,

and (II) the distribution of the winner's survivors,

for fixed-force-level-breakpoint battles. Then we will give analytical results for the following probabilistic versions of the homogeneous-force battle (4.2.1) with stationary transition probabilities corresponding to the time-independent attrition rates (4.7.1):

(a) <u>FT|FT stochastic LANCHESTER-type attrition process</u>

$$A(m,n) = amn \quad \text{and} \quad B(m,n) = bmn , \tag{4.10.1}$$

(b) <u>F|F stochastic LANCHESTER-type attrition process</u>

$$A(m,n) = an \quad \text{and} \quad B(m,n) = bm , \tag{4.10.2}$$

(c) <u>F|FT stochastic LANCHESTER-type attrition process</u>

$$A(m,n) = an \quad \text{and} \quad B(m,n) = bmn . \tag{4.10.3}$$

Finally, we will consider a numerical example to give the reader a feel for the nature of such results.

We begin by developing the fundamental partial-difference equation for the probability of winning for a fixed-force-level-breakpoint battle. Let us denote P[X wins] as $P_X = P_X(m_0,n_0)$, where m_0 and n_0 (as usual) denote the initial numbers of X and Y combatants in the battle. Here, a win for X means that the Y force reached its breakpoint first (see Chapter 3 for further details), i.e. $N(t_f) = n_{BP}$ at the end of battle but $M(t) > m_{BP}$ throughout the battle for $0 \le t \le t_f$. To develop the fundamental equation for $P_X(m_0,n_0)$, we consider the event that X wins a battle in which the initial X force level is m_0 and that of Y is n_0. Let the next casualty occur and consider what X must do in order to win:

1. if an X casualty has occurred, then X must win the remaining battle in which his initial force level is (m_0-1) and that of Y is n_0;

2. if a Y casualty has occurred, then X must win the remaining battle in which his initial force level is m_0 and that of Y is (n_0-1).

Since these events are mutually exclusive and exhaustive, by the theorem of total probability it follows that

$$P_X(m_0,n_0) = P^X_{NC}(m_0,n_0)\ P_X(m_0-1,n_0) + P^Y_{NC}(m_0,n_0)\ P_X(m_0,n_0-1) \ ,$$

where $P^X_{NC}(m_0,n_0) = P[X \text{ casualty}|\text{casualty occurs}]$, which is given by (4.7.14), and similarly for $P^Y_{NC}(m_0,n_0)$. To make our mathematical model "properly posed," i.e. have a well-determined solution (e.g. <u>see</u> COURANT and HILBERT [18, pp. 226-227] for further discussion), we must also specify the appropriate boundary conditions for $P_X(m_0,n_0)$. The natural ones are that X must win if Y starts at his breakpoint, i.e. $P_X(m_0,n_{BP}) = 1$ for $m_0 > m_{BP}$, and that Y must win if X starts at his, i.e. $P_X(m_{BP},n_0) = 0$ for $n_0 > n_{BP}$. Thus, the <u>fundamental partial-difference equation satisfied by the probability of</u> X <u>winning</u> $P_X(m_0,n_0)$ <u>in a fixed-force-level-breakpoint battle</u> is given by (for $m_0 > m_{BP}$ and $n_0 > n_{BP}$)

$$P_X(m_0,n_0) = P^X_{NC}(m_0,n_0)\ P_X(m_0-1,n_0) + P^Y_{NC}(m_0,n_0)\ P_X(m_0,n_0-1) \ , \tag{4.10.4}$$

with boundary conditions

$$P_X(m_0,n_{BP}) = 1 \qquad \text{and} \qquad P_X(m_{BP},n_0) = 0 \ . \tag{4.10.5}$$

The initial state space and boundary conditions are shown in Figure 4.17. Since we have assumed [see assumption (A3) of Section 4.3] that casualties can only occur singly when they do occur, i.e. P[more than one casualty in short time interval of length Δt] = $O((\Delta t)^2)$ or P[more than one casualty at a time] = 0, it is impossible to have the battle end in a draw and then the probability that Y wins $P_Y(m_0,n_0)$ is given by $P_Y(m_0,n_0) = 1 - P_X(m_0,n_0)$.

The probability distribution for the winner's survivors satisfies a similar partial-difference equation. Let us denote P[X wins and has m survivors] as $P_{m,n_{BP}}(m_0,n_0)$. To be more precise, $P_{m,n_{BP}}(m_0,n_0)$ really denotes P[at some time during the battle there are m X survivors and n_{BP} for Y], and the fact that we are considering a fixed-force-level-breakpoint battle (with no replacements and no withdrawals) yields that this probabilty is equivalent to P[X wins and has m survivors]. [If we were to have need for it, $P_{m_{BP},n}(m_0,n_0)$ would be similarly defined.] Then, arguments similar to those used above yield the following fundamental partial-difference equation for the probability that X wins with m final survivors $P_{m,n_{BP}}(m_0,n_0)$ in a fixed-force-level-breakpoint battle is given by (for $m_0 \geq m > m_{BP}$ and $n_0 > n_{BP}$)

$$P_{m,n_{BP}}(m_0,n_0) = P^X_{NC}(m_0,n_0)\, P_{m,n_{BP}}(m_0-1,n_0) + P^Y_{NC}(m_0,n_0)\, P_{m,n_{BP}}(m_0,n_0-1)\,, \qquad (4.10.6)$$

with boundary conditions

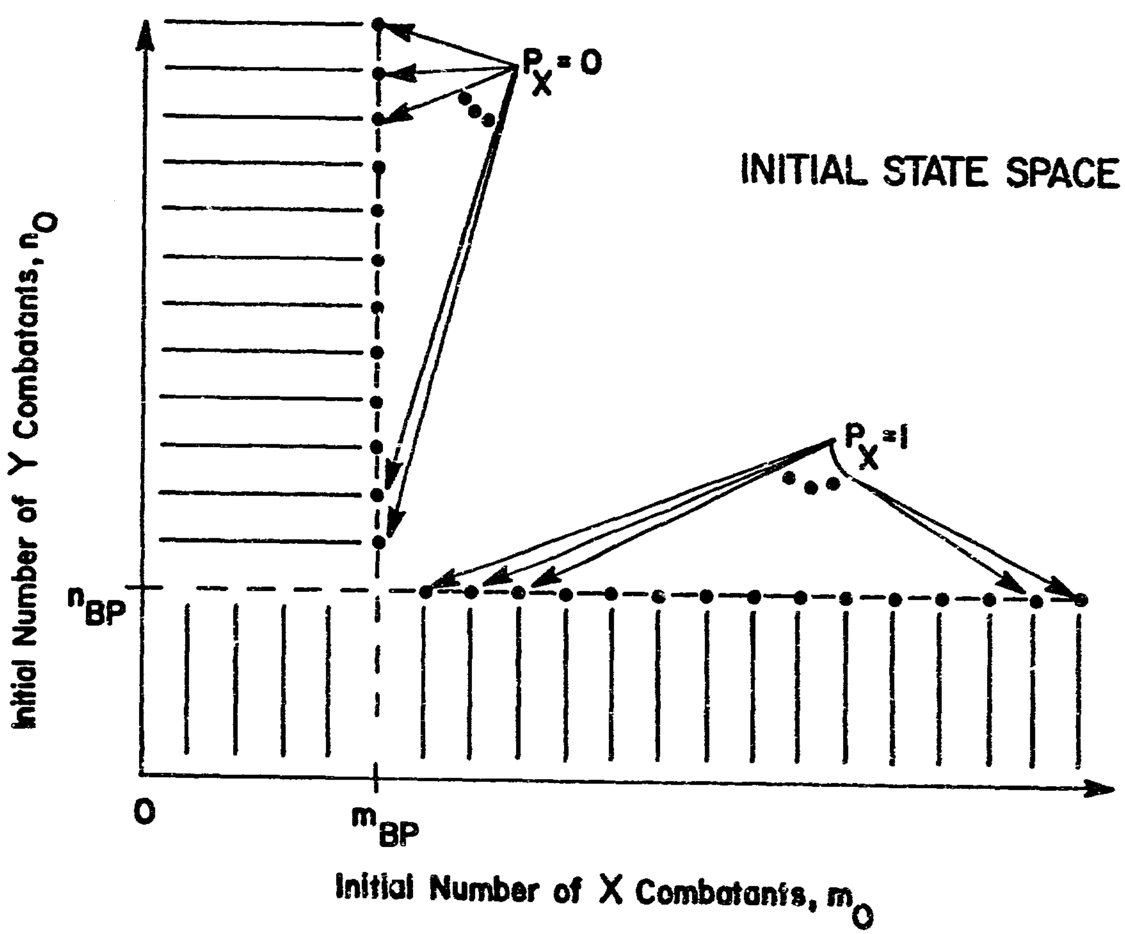


Figure 4.17. The initial state space and boundary conditions for the fundamental partial-difference equation (4.10.4) satisfied by the probability of winning $P_X(m_0,n_0)$ in a fixed-force-level breakpoint battle. Here both the initial state space and also the boundary conditions are only well defined for the appropriate nonnegative integer pairs of the form (m_0,n_0).

$$P_{m,n_{BP}}(m_0,n_{BP}) = \begin{cases} 1 & \text{for } m_0 = m > m_{BP}, \\ 0 & \text{otherwise} \end{cases}$$

and (4.10.7)

$$P_{m,n_{BP}}(m-1,n_0) = 0 \quad \text{for } n_0 \geq n_{BP}.$$

The initial state space and boundary conditions are shown in Figure 4.16. It should be noted that the partial-difference equation (4.10.6) is just a special case of (4.9.23), but that the boundary conditions for the former (4.10.7) are not a special case of those for the latter (4.9.24). We now also observe that

$$P_X(m_0,n_0) = \sum_{m=m_{BP}+1}^{m_0} P_{m,n_{BP}}(m_0,n_0) , \tag{4.10.8}$$

whence follow (4.10.4) and (4.10.5) from (4.10.6) and (4.10.7). We will now examine what analytical results for $P_X(m_0,n_0)$ and $P_{m,n_{BP}}(m_0,n_0)$ have been obtained for each of the above three attrition processes (4.10.1) through (4.10.3).

For the FT|FT stochastic LANCHESTER-type attrition process with attrition rates (4.10.1), one finds that[20] (see MORSE and KIMBALL [65, pp. 67-68], BROWN [15], G. H. WEISS [89], and SMITH [75] for further details)

$$P_{m,n_{BP}} = \binom{m_0+n_0-m-n_{BP}-1}{n_0-n_{BP}-1} \left(\frac{a}{a+b}\right)^{m_0-m} \left(\frac{b}{a+b}\right)^{n_0-n_{BP}} , \tag{4.10.9}$$

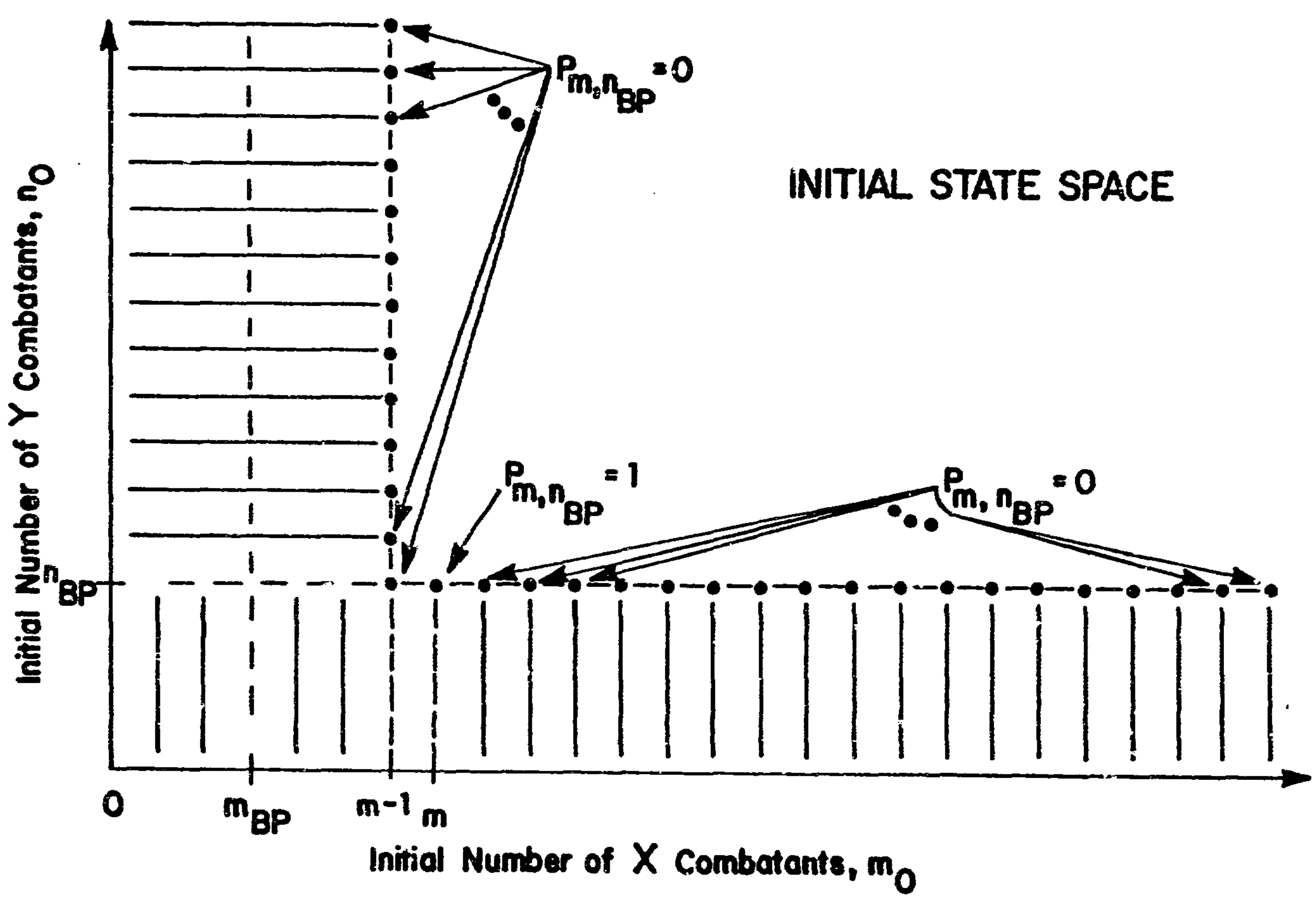


Figure 4.18. The initial state space and boundary conditions for the fundamental partial-difference equation (4.10.6) satisfied by the probability that X wins and has m survivors $P_{m,n_{BP}}(m_0,n_0)$ in a fixed-force-level-breakpoint battle. Here both the initial state space and also the boundary conditions are only well defined for the appropriate nonnegative integer pairs of the form (m_0,n_0).

$$P_{m_{BP},n} = \binom{m_0+n_0-m_{BP}-n-1}{m_0-m_{BP}-1}\left(\frac{a}{a+b}\right)^{m_0-m_{BP}}\left(\frac{b}{a+b}\right)^{n_0-n} , \qquad (4.10.10)$$

$$P_X = \left(\frac{b}{a+b}\right)^{n_0-n_{BP}} \sum_{J=0}^{m_0-m_{BP}-1} \binom{n_0-n_{BP}+J-1}{J}\left(\frac{a}{a+b}\right)^{J} , \qquad (4.10.11)$$

$$P_Y = \left(\frac{a}{a+b}\right)^{m_0-m_{BP}} \sum_{K=0}^{n_0-n_{BP}-1} \binom{m_0-m_{BP}+K-1}{K}\left(\frac{b}{a+b}\right)^{K} . \qquad (4.10.12)$$

Here, for example, $P_{m,n_{BP}}(m_0,n_0)$ as given by (4.10.9) is the solution to the following partial-difference equation for $m_0 \geq m > m_{BP}$ and $n_0 > n_{BP}$

$$P_{m,n_{BP}}(m_0,n_0) = \left(\frac{a}{a+b}\right) P_{m,n_{BP}}(m_0-1,n_0) + \left(\frac{b}{a+b}\right) P_{m,n_{BP}}(m_0,n_0-1) , \qquad (4.10.13)$$

with boundary conditions (4.10.7), since $P^X_{NC} = (m_0,n_0)$ $A(m_0,n_0)/\{A(m_0,n_0) + B(m_0,n_0)\}$ and similarly for $P^Y_{NC}(m_0,n_0)$. The solution (4.10.9) to this partial-difference equation (4.10.13) with boundary conditions (4.10.7) is developed by the method of generating functions below in Appendix C. However, for the FT|FT stochastic LANCHESTER-type battle, simple probabilistic arguments[21] may also be used to obtain (4.10.9). Using the results of PEARSON [70], one may show that (see G. H. WEISS [89] for further details)

$$P_X(m_0,n_0) = I_\beta(n_0 - n_{BP}, m_0 - m_{BP}) , \qquad (4.10.14)$$

where

$$\beta = \frac{b}{a + b} \quad , \tag{4.10.15}$$

$I_x(a,b)$ denotes the incomplete beta function, which may be defined by

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1} (1-t)^{b-1}\, dt \ , \tag{4.10.16}$$

and $B(a,b)$ denotes the usual beta function defined by

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1}\, dt \ . \tag{4.10.17}$$

Tables of the incomplete beta function (equivalently, cumulative binomial probability distribution) are fairly readily available (e.g. see PEARSON [71] or WEINTRAUB [88]; see also further tables listed in ABRAMOWITZ and STEGUN [1, p. 963]). We also have the conditional distribution

$$P[M_f = m \mid X \text{ wins}] = \frac{\binom{m_0+n_0-m-n_{BP}-1}{n_0-n_{BP}-1}}{I_\beta(n_0-n_{BP}, m_0-m_{BP})} \left(\frac{a}{a+b}\right)^{m_0-m} \left(\frac{b}{a+b}\right)^{n_0-n_{BP}} \ , \tag{4.10.18}$$

and the conditional expectation $\bar{m}_X = \sum_{m=m_{BP}+1}^{m_0} mP[M_f = m \mid X \text{ wins}]$ given by (see G. H. WEISS [89] for further details)

$$\bar{m}_X = m_0 - m_{BP} - R(n_0 - n_{BP})$$
$$+ (m_0-m_{BP}) \binom{m_0+n_0-m_{BP}-1}{n_0-n_{BP}-1} \frac{R^{m_0-m_{NP}}}{(1+R)^{m_0+n_0-m_{BP}-n_{BP}-1}} \ , \tag{4.10.19}$$

where (as usual)

$$R = a/b \ . \tag{4.10.20}$$

For the <u>F|F stochastic LANCHESTER-type attrition process with attrition rates</u> (4.10.2), one finds that (<u>see</u> BROWN [15] and SMITH [75] for futher details)

$$P_{m,n_{BP}} = m\left(\frac{b}{a}\right)^{n_0-n_{BP}} \sum_{j=m}^{m_0} \frac{(-1)^{m_0-j}\, j^{m_0+n_0-m-n_{BP}-1}\, \Gamma(\frac{b}{a}j + n_{BP} + 1)}{(m_0 - j)!\,(j - m)!\,\Gamma(\frac{b}{a}j + n_0 + 1)} \ , \tag{4.10.21}$$

$$P_X = \left(\frac{b}{a}\right)^{n_0-n_{BP}} \sum_{j=m_{BP+1}}^{m_0} \frac{(-1)^{m_0-j}\, j^{m_0+n_0-m_{BP}-n_{BP}-1}\, \Gamma(\frac{b}{a}j + n_{BP} + 1)}{(m_0 - j)!\,(j - M_{BP} - 1)!\,\Gamma(\frac{b}{a}j + n_0 + 1)} , \tag{4.10.22}$$

with the corresponding expressions for $P_{m_{BP},n}$ and P_Y being symmetric to the above results. SMITH's [75] results for $P_{m,n_{BP}}(m_0,n_0)$ and $P_X(m_0,n_0)$ with $m_{BP} = n_{BP} = 0$ have been subsequently rediscovered by GYE and Lewis [34] and extended to results equivalent to (4.10.21) and (4.10.22) by GOLDIE [31]. Here, for example, $P_{m,n_{BP}}(m_0,n_0)$ as given by (4.10.21) is the solution to the following partial-difference equation for $m_0 \geq m > m_{BP}$ and $n_0 > n_{BP}$

$$P_{m,n_{BP}}(m_0,n_0) = \left(\frac{an_0}{bm_0 + an_0}\right) P_{m,n_{BP}}(m_0-1,n_0) + \left(\frac{bm_0}{bm_0 + an_0}\right) P_{m,n_{BP}}(m_0,\ n_0-1) \ , \tag{4.10.23}$$

with boundary conditions (4.10.5), since $P^X_{NC}(m_0,n_0) = A(m_0,n_0)/\{A(m_0,n_0) + B(m_0,n_0)\}$ and $P^Y_{NC}(m_0,n_0) = 1 - P^X_{NC}$. The solution (4.10.21) to this partial-difference equation (4.10.23) with boundary conditions (4.10.7) is developed by BROWN's separation-of-variables method below in Appendix C. In obtaining P_X from $P_{m,n_{BP}}$, one makes use of the facts that[22]

$$\sum_{m=m_{BP}+1}^{j} \frac{m j^{-m}}{(j-m)!} = \frac{j^{-m_{BP}}}{(j-m_{BP}-1)!}, \tag{4.10.24}$$

and also that $1/(j-m)! = 1/\Gamma(j-m+1) = 0$ for all integers $j < m$.

For the *F|FT stochastic LANCHESTER-type attrition process with attrition rates* (4.10.3), one finds that (*see* SMITH [75] or KISI and HIROSE [53] for further details)

$$P_{m,n_{BP}} = m \sum_{j=m}^{m_0} \frac{(-1)^{m_0-j}\, j^{m_0+n_0-m-n_{BP}-1}}{(m_0-j)!\,(j-m)!\,(j+a/b)^{n_0-n_{BP}}}, \tag{4.10.25}$$

$$P_{m_{BP},n} = \left(\frac{a}{b}\right)\binom{m_0}{m_{BP}} \sum_{k=m_{BP}+1}^{m_0} \frac{(-1)^{m_0-k}\, k^{m_0+n_0-m_{BP}-n-1}}{(m_0-k)!\,(k-m_{BP}-1)!\,(k+a/b)^{n_0-n+1}}, \tag{4.10.26}$$

$$P_X = \sum_{j=m_{BP}+1}^{m_0} \frac{(-1)^{m_0-j}\, j^{m_0+n_0-m_{BP}-n_{BP}-1}}{(m_0-j)!\,(j-m_{BP}-1)!\,(j+a/b)^{n_0-n_{BP}}}, \tag{4.10.27}$$

$$P_Y = \binom{m_0}{m_{BP}} \sum_{k=m_{BP}+1}^{m_0} \frac{(-1)^{m_0-k}\, k^{m_0-m_{BP}-1}\{(k+a/b)^{n_0-n_{BP}} - k^{n_0-n_{BP}}\}}{(m_0-k)!\,(k-m_{BP}-1)!\,(k+a/b)^{n_0-n_{BP}}}, \tag{4.10.28}$$

Here, for example, the solution (4.10.25) to the fundamental partial-difference equation for $P_{m,n_{BP}}(m_0,n_0)$ may be developed by BROWN's separation-of-variables method (<u>see</u> Appendix C). Again, to obtain (for example) (4.10.27) from (4.10.25) one uses (4.10.24) and the fact that $1/(j-m)! = 1/\Gamma(j-m+1) = 0$ for all integers $j < m$.

In Figure 4.19 we have plotted for the FT|FT attrition process and a fight to the finish P_Y versus the quantity $bm_0/(an_0)$, which the reader may think of as the deterministic-battle-outcome-prediction variable. Although (strictly speaking) for fixed n_0 and a given bound on m_0 the dependent variable P_Y is only defined for a finite set of values of the independent variable $bm_0/(an_0)$, we have taken the liberty of drawing "continuously-connected" curves. The reader can see that P_Y depends on the absolute numbers of initial combatants in the battle and that $P_Y \to$ (deterministic-battle-outcome-prediction result) as $n_0 \to +\infty$ if we give the appropriate probabilistic interpretation to the force-annihilation-prediction condition (i.e. $P_Y = 1.0$ for $x_0/y_0 < a/b$). Thus, the deterministic-battle-outcome-prediction result may be considered to be a step function when the probability of winning is plotted against the appropriate measure of parity between the two opposing forces. The probability of winning as a function of this measure of parity asymptotically approaches this step function as the initial number of combatants becomes arbitrarily large. This same type of behavior also holds for the other two stochastic LANCHESTER-type attrition processes considered above (i.e. the F|F and F|FT stochastic attrition process). In all the cases known to this author, the slope of the curve of the probability of winning

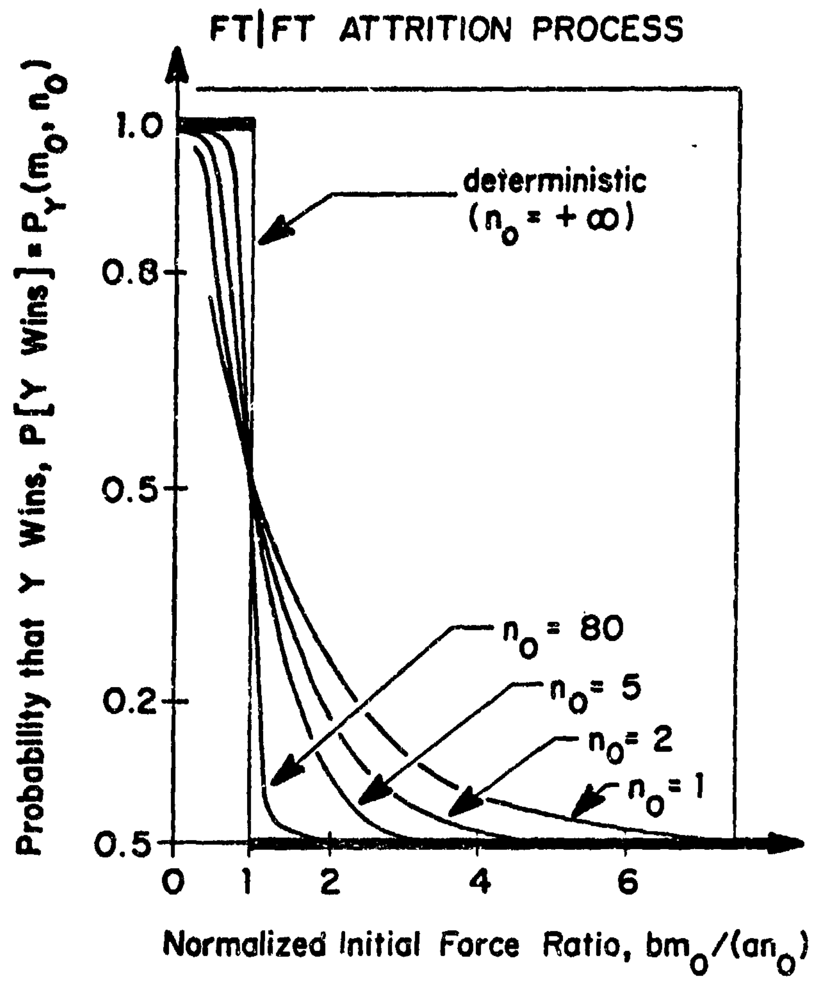


Figure 4.19. The probability that Y wins P_Y as a function of the normalized initial force ratio $bm_0/(an_0)$ for the FT|FT attrition process and a fight to the finish. Shown here are curves for $n_0 = 1, 2, 5,$ and 80 and also the corresponding deterministic-battle-outcome-prediction result which corresponds to $n_0 = +\infty$. For the calculations shown here we have taken $a = b$, and consequently P[X casualty|casualty occurs] = 0.5.

versus the appropriate measure of force parity is steepest at the point of parity between the forces. In other words, at parity the addition of one more combatant initially to battle has its greatest impact on the outcome of the battle (as quantified by the probability of winning) (*see* LEE and WANNASILPA [57] or CRAIG [19] for many additional such plots of the probability of winning versus some measure of force parity). Here it has seemed appropriate to say that two forces are at parity in such a stochastic battle if either is equally likely to win (i.e. $P_X = P_Y = 0.5$).

4.11. Approximations to the Probability of Winning.

As the results of the previous section show, the exact analytical expression for the probability that a given side will win is far too complicated to be of practical use[23]. Moreover, if one tries to use such an exact analytical expression for computation on, for example, a large-scale digital computer of results for a battle with any appreciable numbers of initial combatants, one finds that the attempt to compute a factorial quantity such as $(m_0 + n_0)!$ or a power quantity such as $m_0^{(m_0+n_0)}$ causes all sorts of numerical problems[24]. To avoid such numerical problems, one can try to recursively compute such quantities, and after much involved labor along these lines, one finds out that he has rediscovered the fundamental partial-difference equation that gave rise to the exact analytical results in the first place. In other words, it is easier to use the fundamental partial-difference equation directly in a numerical algorithm than to use the exact analytical results for the probability of winning or the distribution of the winner's final survivors (see CRAIG [19, p. 26 and pp. 43-52] for further details). Because of these computational shortcomings of exact results, one must rely on approximations instead of exact analytical results to develop insights into how the distribution of battle outcomes is related to the initial numbers of combatants and the probabilistic combat dynamics[25]. What is needed is a simple approximation that will enable one to perceive the role played by the combatants' attrition rates $A(m,n)$ and $B(m,n)$ and initial numbers in determining the probability of winning.

Thus, in this section we will give some simple approximations to the probability of winning for the FT|FT, F|F, and F|FT attrtion processes

with attrition rates (4.10.1) through (4.10.3) for which we have developed exact analytical results in the previous section. These results are given for fixed-force-level-breakpoint battles except for the F|F attrition process for which results are given only for a fight to the finish. All the approximations given in this section for the stochastic battles of the previous section are essentially of the form

$$\hat{P}_X = \Phi(v) , \qquad (4.11.1)$$

where P_X denotes the approximation to $\hat{P}_X(m_0,n_0)$, $\Phi(v)$ denotes the cumulative distribution function (c.d.f.) of the unit (or standardized) normal deviate, i.e.

$$\Phi(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} e^{-t^2/2} \, dt , \qquad (4.11.2)$$

and the argument v depends on the type of attrition process and the battle-termination conditions. However, KISI and HIROSE [53] have given a POISSON approximation (see also SPRINGALL [77, pp. 133-136 and pp. 167-171]) for the probability of winning in the F|FT battle (4.10.3).

The great value of the normal approximation (4.11.1), however, is that the c.d.f. of the unit normal is so extremely well known and tables (and also computer routines) are readily available (e.g. ABRAMOWITZ and STEGUN [1, pp. 966-972]). Furthermore, we will see that as the initial force ratio $u_0 = m_0/n_0$ varies between 0 and $+\infty$, the argument v in (4.11.1) varies between $-\infty$ and $+\infty$, and we may therefore invoke under the appropriate

conditions the following asymptotic approximations based on the well-known simple asymptotic approximations to the c.d.f. for the normal distribution (e.g. see FELLER [25, p. 166])

I. for $v \to +\infty$:

$$\hat{P}_X \sim 1 - \frac{1}{v\sqrt{2\pi}} e^{-v^2/2} , \qquad (4.11.3)$$

II. for $v \to -\infty$:

$$\hat{P}_X \sim \frac{1}{|v|\sqrt{2\pi}} e^{-v^2/2} , \qquad (4.11.4)$$

where the symbol $\sim$ is used to indicate that the ratio of the two sides tends to one under the stated limiting condition.

Before we consider the particulars of the approximations that have been developed, the author would like to point out to the reader the following shortcomings of this work:

(S1) no a priori error bounds exist,

(S2) no general method is known for developing such approximations.

With respect to this latter shortcoming (S2), BROWN [15] (see also BROWN [14]) has given an approach that might contain the germ of an idea for developing the desired unified approach. Let us now examine what approximations to the

probability of winning have been developed for the above three attrition processes with attrition rates (4.10.1) through (4.10.3) for a fixed-force-level-breakpoint battle[26]. For each of these battle types, we will denote the approximation to the probability of winning as $\hat{P}_X$.

For the FT|FT stochastic LANCHESTER-type attrition process with attrition rates (4.10.1) and a fixed-force-level-breakpoint battle, one may develop (see BROWN [14; 15] and G. H. WEISS [89] for further details[27]) the approximation (4.11.1) with argument v given by

$$v = \sqrt{\frac{n_0(1 - f_{BP}^X)}{R}} \left\{ \frac{u_0 - f_\rho^c R}{\sqrt{u_0 + f_\rho^c}} \right\} , \tag{4.11.5}$$

where $R = a/b$, $u_0 = m_0/n_0$, $f_\rho^c = (1 - f_{BP}^Y)/(1 - f_{BP}^X)$, and the breakpoints are (as usual) expressed in the form $x_{BP} = f_{BP}^X x_0$ and $y_{BP} = f_{BP}^Y y_0$. It is worthwhile noting the following special case: $f_{BP}^X = f_{BP}^Y = 0$ and $a = b$. In this case (4.11.5) reduces to

$$v = \frac{m_0 - n_0}{\sqrt{m_0 + n_0}} ,$$

which clearly shows that the probability of winning is dependent upon the total number of combatants in the battle except when parity exists between the forces and $P_X = 0.5$. There are "better" choices for v in the sense that they give closer approximations (see FELLER [23]), but the above choice (made by BROWN [15]) has the merit of simplicity. The above approximation (4.11.1) with argument v given by (4.11.5) follows from BROWN's [15, p. 422] result for a fight to the finish and the observation that the exact result for a fixed-force-level-breakpoint battle (4.10.14) may be obtained from that for a fight to the finish by replacing m_0 by $(m_0 - m_{BP})$ and n_0

by $(n_0 - n_{BP})$, whence follows (4.11.1) with argument v given by

$$v = \frac{b(m_0 - m_{BP}) - a(n_0 - n_{BP})}{\sqrt{ab(m_0 - m_{BP} + n_0 - n_{BP})}} , \qquad (4.11.6)$$

which is equivalent to (4.11.5).

For the <u>F|F stochastic LANCHESTER-type attrition process with attrition rates</u> (4.10.2) <u>and a fight to the finish</u>, BROWN [14; 15] has developed the approximation (4.11.1) with argument v given by

$$v = \sqrt{\frac{3n_0}{\sqrt{R}}} \left\{ \frac{u_0 - \sqrt{R}}{\sqrt{u_0 + 1}} \right\} . \qquad (4.11.7)$$

Again, (4.11.7) shows us that the probability of winning (at least according to the above approximation) is dependent on the total number of combatants in the battle except when parity exists between the forces with $u_0 = \sqrt{R}$.

For the <u>F|FT stochastic LANCHESTER-type attrition process with attrition rates</u> (4.10.3) <u>and a fixed-force-level-breakpoint battle</u>, KISE and HIROSE [53] have developed the following POISSON approximation from consideration of a generating function

$$\hat{P}'_X = 1 - \sum_{j=0}^{n_0 - n_{BP} - 1} \left(\frac{q^j}{j!} \right) e^{-q} , \qquad (4.11.8)$$

where $q = (m_0^2 - m_{BP}^2)/(2R)$ and $R = a/b$. However, since the POISSON and chi-square (or χ^2) distributions are related (e.g. <u>see</u> PARZEN [68, p. 178

and p. 181] or ABRAMOWITZ and STEGUN [1, p. 941]), we may also write

$$\hat{P}_{X'} = 1 - Q(2q,\ 2\{n_0 - n_{BP}\}) \ , \tag{4.11.9}$$

where $Q(\chi^2|\nu)$ denotes the complementary cumulative distribution function for the χ^2 distribution with ν degrees of freedom. Although (4.11.8) and (4.11.9) are, of course, entirely equivalent, the latter result is somewhat more significant, since not only are more tables available for the χ^2 distribution but also there are well-known normal approximations to it (e.g. see KENDALL [52, p. 294] and ABRAMOWITZ and STEGUN [1, p. 941]). Thus, we may use normal approximations to the χ^2 distribution to obtain further (and in some sense simpler) approximations to the probability of winning: namely,

$$\hat{P}_X = \Phi(v_i) \ , \tag{4.11.10}$$

where

(I) for $n_0 - n_{BP} > 50$, $i = 1$ and then the argument v_1 is given by

$$v_1 = 2\left\{\sqrt{\frac{m_0^2 - m_{BP}^2}{2R}} - \sqrt{n_0 - n_{BP} - 0.25}\right\} \ ; \tag{4.11.11}$$

(II) for $n_0 - n_{BP} > 15$, $i = 2$ and then the argument v_2 is given by

$$v_2 = \frac{1}{\varepsilon}\left\{\left[\frac{m_0^2 - m_{BP}^2}{2R(n_0 - n_{BP})}\right]^{1/3} - (1 - \varepsilon^2)\right\} \ , \tag{4.11.12}$$

with

$$\varepsilon = 1/(3\sqrt{n_0 - n_{BP}}) \ .$$

The above regions of applicability for the approximation (4.11.10) are based on conditions given by ABRAMOWITZ and STEGUN [1, p. 941] for the normal approximation to the χ^2. These limits for the regions of the approximation's applicability may be very conservative, and in practice one may be able to use (4.11.11) and (4.11.12) for the values of $(n_0 - n_{BP})$ as small as 10. Along these lines, it will be instructive to consider a numerical example due to KISE and HIROSE [53]: let $m_0 = 100$, $n_0 = 10$, $R = 500$, and $m_{BP} = n_{BP} = 0$. Then one finds that

$$P_X(100, 10) = 0.5460$$

$$\hat{P}'_X = 0.5421 \qquad \text{from (4.11.9)}$$

$$\hat{P}_X(v_1) = 0.5317 \qquad \text{from (4.11.10) and (4.11.11),}$$

and

$$\hat{P}_X(v_2) = 0.5420 \qquad \text{from (4.11.10) and (4.11.12).}$$

Thus (at least in this one specific example), the above normal approximation (4.11.10) with argument given by either (4.11.11) or (4.11.12) are very good (less than 2 percent error) for even $n_0 - n_{BP} = 10$, with the more complicated approximation (4.11.12) being more accurate (less than 1 percent error).

4.12. The Average Force Levels.

As we have seen above in Section 4.9, the joint probability distribution for the numbers of survivors is not a very enlightening measure of a battle's progress for even rather modest numbers of combatants because of its inherent complexity in terms of number of components. Most decision makers and many practical analysts prefer one number to represent the military strength of each of the two opposing forces. One such obvious number of interest to the military analyst (cf. Table 4.I) is the average number of combatants on each side (here assumed to be homogeneous). One is also interested in the variability in the mean course of combat (i.e. the dispersion of the number of survivors about its mean value) in order to gauge the risk in using these mean values to represent the probabilistic evolution of combat. Thus, in this section we will consider the average force levels, while in the next one we will examine the variance and covariance (e.g. see PARZEN [68, p. 356]) of the force levels. These quantities are related to the first two moments of the force levels, and for purposes of discussing their numerical computation, it is convenient to first discuss the general computation of force-level moments.

There are essentially two methods for computing the moments of each side's force level:

(MM1') compute them directly from the joint distribution of the numbers of survivors [i.e. from $P(t,m,n)$],

or (MM2') compute them by first determining the differential equation satisfied by the moment under consideration and then solving this equation.

For convenience, we will refer to these two basic moment-calculation methods simply as follows:

(MM1) direct-computation method,

and (MM2) moment-differential-equation method.

The direct-computation method (MM1) uses the joint probability distribution for the numbers of survivors, i.e. $P(t,m,n)$ for $m_{BP} \leq m \leq m_0$ and $n_{BP} \leq n \leq n_0$, to compute the moment under consideration directly from the definition of mathematical expectation, i.e.

$$E[f(M,N)] = \sum_{m=m_{BP}}^{m_0} \sum_{n=n_{BP}}^{n_0} f(m,n)\, P(t,m,n) \ .$$

Consequently, one must have previously determined $P(t,m,n)$ to use the direct-computation method. As we saw in Section 4.9, there are basically three methods for computing the distribution of survivors (i.e. the state probability vector): (M1) the analytical method, (M2) the numerical method, and (M3) the hybrid analytical-numerical method. After $P(t,m,n)$ has been numerically determined by one of these three computational methods, one can simply compute the desired moment directly from its definition. On the other hand, the moment-differential-equation method (MM2) is completely different in seeking to determine an equation for the rate of change of the force-level moment under consideration by using, for example, the forward KOLMOGOROV equations, e.g. (4.7.2) through (4.7.8). The basic idea behind this method is to be able

to solve the resulting moment differential equation (or system of equations if the moment under consideration cannot be decoupled from others) for the sought quantity. Unfortunately (as we will see below), one still needs to know $P(t,m,n)$ to be able to solve the moment differential equation, but one can make some rough approximations to eliminate this requirement and simplify this method. We will now examine these two moment-calculation methods (MM1) and (MM2) further, with emphasis being given to the second one (MM2) for the calculation of the average force levels.

Direct computation of the average force levels, i.e. method (MM1), is straightforward and merely involves computing

$$\bar{m}(t) = E[M(t)] = \sum_{m=m_{BP}}^{m_0} \sum_{n=n_{BP}}^{n_0} mP(t,m,n) \ , \tag{4.12.1}$$

and

$$\bar{n}(t) = E[N(t)] = \sum_{m=m_{BP}}^{m_0} \sum_{n=n_{BP}}^{n_0} nP(t,m,n) \ , \tag{4.12.2}$$

where $\bar{m}(t)$ denotes the average X force level at time t and similarly for $\bar{n}(t)$. As already mentioned above, one must know $P(t,m,n)$ for $m_{BP} \leq m \leq m_0$ and $n_{BP} \leq n \leq n_0$ in order to use this force-level-moment-calculation method, and we have previously discussed in Section 4.9 three methods for numerically determining $P(t,m,n)$. One point that does merit further discussion, however, is the tremendous computational advantage in using CLARK's hybrid analytical-numerical method for such calculations, for computing not only the joint probability distribution for the numbers of survivors but also the moments of each side's force level (including both the average force levels and also

their variability). G. CLARK [16, pp. 112-114] has shown that the $i^{\underline{th}}$ moment of, for example, the X force level may be computed as follows

$$E[M^i(t)] = D_{0,0}^{(i)} + \sum_{j=1}^{m_0} \sum_{k=1}^{n_0} D_{j,k}^{(i)} \exp[-\{A(j,k) + B(j,k)\}t] \,, \tag{4.12.3}$$

where a coefficient such as $D_{j,k}^{(i)}$ is the $i^{\underline{th}}$ "incomplete moment" of the $C_{j,k}^{m,n}$ coefficients from CLARK's hybrid expression for the state probabilities. More specifically, the moment coefficients $D_{j,k}^{(i)}$ for $1 \leq j \leq m_0$ and $1 \leq k \leq n_0$ are given by

$$D_{j,k}^{(i)} = \sum_{m=1}^{j} \sum_{n=0}^{k} m^i \, C_{j,k}^{m,n} \,, \tag{4.12.4}$$

and

$$D_{0,0}^{(i)} = \sum_{m=1}^{m_0} m^i \, C_{0,0}^{0,n} \tag{4.12.5}$$

where the coefficients $C_{j,k}^{m,n}$ are given by (4.9.5) through (4.9.10). The great computational advantage in computing the $i^{\underline{th}}$ moment from the analytical expression (4.12.3) with the $D_{j,k}^{(i)}$ coefficients numerically determined by (4.12.4) and (4.12.5) lies in the facts that (1) these coefficients are simply and easily computed from the numerical results for the $C_{j,k}^{m,n}$ coefficients of the state probabilities and (2) they need only be computed <u>once</u> for a given set of battle parameters. Thus, this hybrid analytical-numerical method is very efficient (in fact, over 50 times faster than using exact analytical results according to an example reported by CLARK [16, p. 115]) for computing time histories of the moments.

Let us now turn to the moment-differential-equation method (MM2) for computing the average force levels for the general homogeneous-force autonomous model given by (4.7.2) through (4.7.8). We will see that although this method is not at all useful for directly computing exact values of the average force levels, it does provide considerable insight into the behavior of, for example, X's average force level, which (however) is much more efficiently computed from (4.12.3) with $i = 1$. For the general model given by (4.7.2) through (4.7.8) we find that

$$\begin{cases} \frac{d}{dt} E[M] = -E[G(t,M,N)] + \Sigma_X(t) \quad \text{with} \quad E[M(0)] = m_0 , \\ \\ \frac{d}{dt} E[N] = -E[H(t,M,N)] + \Sigma_Y(t) \quad \text{with} \quad E[N(0)] = n_0 , \end{cases} \tag{4.12.6}$$

where the boundary-sum terms $\Sigma_X(t)$ and $\Sigma_Y(t) > 0$ are given by

$$\Sigma_X(t) = \sum_{m=m_{BP}+1}^{m_0} G(t,m,n_{BP})\ P(t,m,n_{BP}) + \sum_{n=n_{BP}+1}^{n_0} G(t,m_{BP},n)\ P(t,m_{BP},n) , \tag{4.12.7}$$

and

$$\Sigma_Y(t) = \sum_{m=m_{BP}+1}^{m_0} H(t,m,n_{BP})\ P(t,m,n_{BP}) + \sum_{n=n_{BP}+1}^{n_0} H(t,m_{BP},n)\ P(t,m_{BP},n) . \tag{4.12.8}$$

These terms arise from accumulation of probability on the boundary of the state space as the system evolves over time and reaches its terminal state. In the steady (i.e. in the long run) all probability has accumulated on the state-space boundary and the average force levels reach their strictly positive limiting values. The boundary-sum terms perform for this behavior the attendant bookkeeping in the expression (4.12.6) for the average force levels.

Let us now sketch the derivation of (4.12.6). It suffices to consider the first equation of (4.12.6). Using (4.5.6) with $g(M) = M$ and $h(N) = 1$, we find that

$$\frac{d}{dt} E[M] = - \sum_{m=m_{BP}+1}^{m_0} \sum_{n=n_{BP}+1}^{n_0} G(t,m,n) P(t,m,n) . \tag{4.12.9}$$

Also, recalling (4.7.8), one can easily show that

$$\begin{aligned} &\sum_{m=m_{BP}+1}^{m_0} \sum_{n=n_{BP}+1}^{n_0} G(t,m,n) P(t,m,n) \\ &= E[G(t,M,N)] - \sum_{m=m_{BP}+1}^{m_0} G(t,m,n_{BP}) P(t,m,n_{BP}) \\ &\qquad - \sum_{n=n_{BP}+1}^{n_0} G(t,m_{BP},n) P(t,m_{BP},) . \end{aligned} \tag{4.12.10}$$

Combination of (4.12.9) and (4.12.10) yields our desired result, the first equation of (4.12.6) with $\Sigma_X(t)$ given by (4.12.7).

<u>Example 4.12.1</u>. For the <u>autonomous stochastic F|F attrition process</u> in which $G(t,m,n) = an$ and $H(t,m,n) = bm$, the average-force-level equations (4.12.6) become

$$\begin{cases} \dfrac{d\bar{m}}{dt} = -a\bar{n} + aS_Y(t) & \text{with } \bar{m}(0) = m_0 , \\ \\ \dfrac{d\bar{n}}{dt} = -b\bar{m} + bS_X(t) & \text{with } \bar{n}(0) = n_0 , \end{cases} \tag{4.12.11}$$

where $\bar{m} = E[M]$, $\bar{n} = E[N]$,

$$S_X(t) = \sum_{m=m_{BP}+1}^{m_0} mP(t,m,n_{BP}) + m_{BP} \sum_{n=n_{BP}+1}^{n_0} P(t,m_{BP},n), \tag{4.12.12}$$

and

$$S_Y(t) = n_{BP} \sum_{m=m_{BP}+1}^{m_0} P(t,m,n_{BP}) + \sum_{n=n_{BP}+1}^{n_0} nP(t,m_{BP},n) . \tag{4.12.13}$$

From their definitions, it is clear that both $S_X(t)$ and $S_Y(t) > 0$ for all $t > 0$. The following interpretation for $S_X(t)$ is worthy of note: $S_X(t)$ is the expected number of X survivors in a battle-terminated state by time t for a fixed-force-level-breakpoint battle. In particular, when $m_{BP} = n_{BP} = 0$, $S_X(t)$ denotes the expected number of X survivors in a battle-terminated state (here, one side or the other annihilated) by time t for a fight to the finish. A similar interpretation applies to $S_Y(t)$. Intuitively we know that both $S_X(t)$ and $S_Y(t)$ are "small," at least as long as there is little chance that the battle has ended by time t. Finally, since all but the terminal battle states are transient, the average force levels approach positive limiting values, and we may accordingly infer the asymptotic behavior

$$S_X(t) \to \bar{m}(t) \to \bar{m}(+\infty) \qquad \text{as} \qquad t \to +\infty , \tag{4.12.14}$$

and similarly for $S_Y(t)$.

Example 4.12.2. For the autonomous stochastic FT|FT attrition process in which $G(t,m,n) = amn$ and $H(t,m,n) = bmn$, the average-force-level equations (4.12.6) become

$$\begin{cases} \frac{d}{dt} E[M] = -aE[MN] + aS(t) \quad \text{with} \quad E[M(0)] = m_0 , \\ \frac{d}{dt} E[N] = -bE[MN] + bS(t) \quad \text{with} \quad E[N(0)] = n_0 , \end{cases} \tag{4.12.15}$$

where $S(t) > 0$ for $t > 0$ is given by

$$S(t) = n_{BP} \sum_{m=m_{BP}+1}^{m_0} mP(t,m,n_{BP}) + m_{BP} \sum_{n=n_{BP}+1}^{n_0} nP(t,m_{BP},n) . \tag{4.12.16}$$

Here, $S(t)$ may be interpreted as the expected value for the product of the numbers of survivors in a battle-terminated state by time t for a fixed-force-level-breakpoint battle.

We will now briefly consider a preliminary theoretical analysis of average force-level behavior based on the above average-force-level differential equations (4.12.6). We will also mention some corroborating numerical investigations.

For the autonomous stochastic F|F attrition process considered in Example 4.12.1 above, let us examine how the average force levels compare with those generated by the corresponding deterministic model with the same attrition-rate coefficients a and b and the same initial numbers of combatants m_0 and n_0. Thus, we consider the corresponding deterministic attrition process

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = m_0 , \\ \dfrac{dy}{dt} = -bx & \text{with } y(0) = n_0 . \end{cases} \tag{4.12.17}$$

Then, if we let

$$\Delta_X = \bar{m} - x, \quad \text{and} \quad \Delta_Y = \bar{n} - y , \tag{4.12.18}$$

it follows that

$$\begin{cases} \dfrac{d}{dt} \Delta_X = -a\Delta_Y + aS_Y(t) & \text{with} \quad \Delta_X(0) = 0 , \\ \dfrac{d}{dt} \Delta_Y = -b\Delta_X + bS_X(t) & \text{with} \quad \Delta_Y(0) = 0 , \end{cases} \tag{4.12.19}$$

which has the solution

$$\Delta_X(t) = \sqrt{ab} \int_0^t \Big\{ S_Y(s) \sqrt{\frac{a}{b}} \cosh[\sqrt{ab}\,(t - s)] - S_X(s) \sinh[\sqrt{ab}\,(t - s)] \Big\} ds , \tag{4.12.20}$$

and

$$\Delta_Y(t) = \sqrt{ab} \int_0^t \Big\{ S_X(s) \sqrt{\frac{b}{a}} \cosh[\sqrt{ab}\,(t - s)] - S_Y(s) \sinh[\sqrt{ab}\,(t - s)] \Big\} ds . \tag{4.12.21}$$

Here, for example, $\Delta_X(t)$ represents the bias in the average force level of the X force in an F|F LANCHESTER-type stochastic attrition process. It measures the departure of the average X force level obtained from this stochastic model from the X force level obtained from the corresponding deterministic model with the same attrition-rate coefficients a and b and the same initial numbers of combatants m_0 and n_0. Thus, $\Delta_X(t) > 0$ means that the stochastic model on the average yields higher force levels than does the corresponding

deterministic model for the same set of input data. From (4.12.20) and (4.12.21), we can identify cases in which one can easily determine the signs of $\Delta_X(t)$ and $\Delta_Y(t)$:

1. X wins very decisively in a fight to the finish.
   In this case $S_Y(t) \approx 0$ all during the battle, and from (4.12.20) and (4.12.21) we see that $\Delta_X(t) < 0$ and $\Delta_Y(t) > 0$ for all $t > 0$. Thus, $\bar{m}(t) < x(t)$ and $\bar{n}(t) > y(t)$ for all $t > 0$.

2. Symmetric parity, i.e. $a = b$, $m_0 = n_0$, and $m_{BP} = n_{BP}$.
   In this case $S_X(t) = S_Y(t) = S(t)$, and from (4.12.20) and (4.12.21) we find that $\Delta_X(t) = \Delta_Y(t) = a \int_0^t S(s) \exp[\sqrt{ab}(t-s)]ds > 0$. Thus, $\bar{m}(t) > x(t)$ and $\bar{n}(t) > y(t)$ for all $t > 0$.

Numerical investigations concerning the above biases have been preformed by CLARK [16] and CRAIG [19]. Figure 4.20 is from CLARK [16] and shows the large biases typically present for small numbers of combatants in the symmetric-parity case just discussed above. CRAIG [19] took CLARK's [16] work as a point of departure and did more extensive numerical investigations. CRAIG [19] computed the average force levels from state probabilities determined by the numerical method (M2), i.e. numerical integration of the forward KOLMOGOROV equations. Based on consideration of many, many specific numerical examples, he formulated the following interesting hypotheses concerning the biases in the average force levels:

(H1) for fixed initial force levels and attrition-rate coefficients, the final force-level biases at the deterministic battle's end decrease as the breakpoint force levels increase (however, as

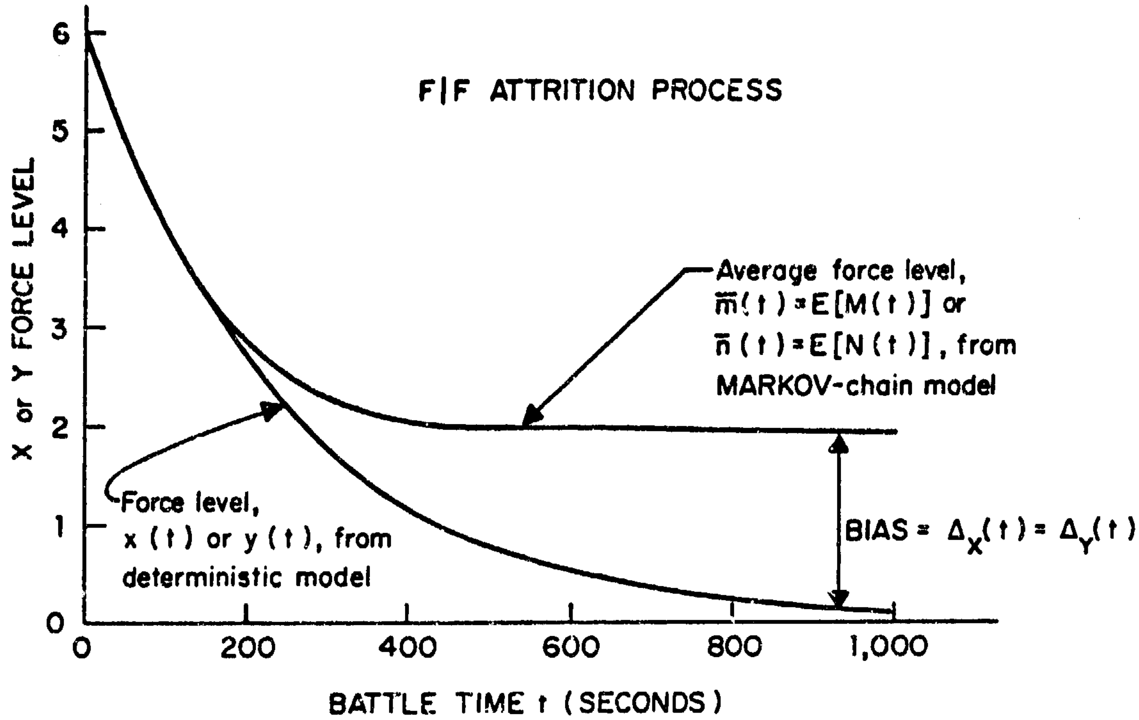


Figure 4.20. Large biases $\Delta_X(t)$ and $\Delta_Y(t)$ in the average force levels in an F|F LANCHESTER-type stochastic attrition process for the symmetric-parity case with small numbers of initial combatants. Here the bias in the average, for example, X force level $\Delta_X(t)$ is defined as $\bar{m}(t) - x(t)$. The input data for this case is
$a = 0.004$ X (casualties/second) per Y firer,
$b = 0.004$ Y (casualties/second) per X firer,
$m_0 = 6$, and $n_0 = 6$.

a percentage of casualties in the deterministic model, the biases increase);

(H2) for everything else equal, the larger the initial force levels become, the larger the numerical biases become in absolute terms but smaller percentagewise;

(H3) the closer the forces come to parity in the deterministic battle, the larger become the biases at the time corresponding to the deterministic battle's end, which itself becomes extended in time;

and (H4) the biases at times corresponding to less than one-half the duration of the deterministic battle are negligible.

For the <u>autonomous stochastic FT|FT attrition process</u> considered in Example 4.12.2 above, we will also theoretically examine the biases $\Delta_X(t)$ and $\Delta_Y(t)$. First, though, let us develop a very interesting result. Multiplying the first of equations (4.12.15) by b, the second by a, and subtracting, we obtain the <u>stochastic linear law</u> for fixed-force-level-breakpoint battles

$$b\bar{m} - a\bar{n} = bm_0 - an_0 \ . \tag{4.12.22}$$

This result is particularly remarkable, since no corresponding simple result holds for the F|F stochastic attrition process. Consider now the corresponding deterministic attrition process for the FT|FT attrition process under consideration

$$\begin{cases} \dfrac{dx}{dt} = -axy & \text{with} \quad x(0) = m_0 \ , \\ \\ \dfrac{dy}{dt} = -bxy & \text{with} \quad y(0) = n_0 \ . \end{cases} \tag{4.12.23}$$

From (4.12.23) we obtain the deterministic linear law $bx - ay = bm_0 - an_0$, whence combination with (4.12.22) yields

$$\Delta_X(t) = \frac{a}{b} \Delta_Y(t) \ , \tag{4.12.24}$$

whence

$$\bar{m}(t) > x(t) \quad \underline{\text{if and only if}} \quad \bar{n}(t) > y(t) \ . \tag{4.12.25}$$

It is interesting to note that even when $m_{BP} = n_{BP} = 0$ and consequently $S(t) \equiv 0$ in (4.12.15), we still have, for example, $\Delta_X(t) \not\equiv 0$, since $E[MN] \neq E[M]\,E[N]$ (<u>see</u> CLARK [16, pp. 81-83] for further details). Numerical investigation of these biases theoretically considered here has been performed by CLARK [16, pp. 116-124]. Some typical results for relatively small numbers of combatants are shown in Figures 4.21 and 4.22, which are from CLARK [16].

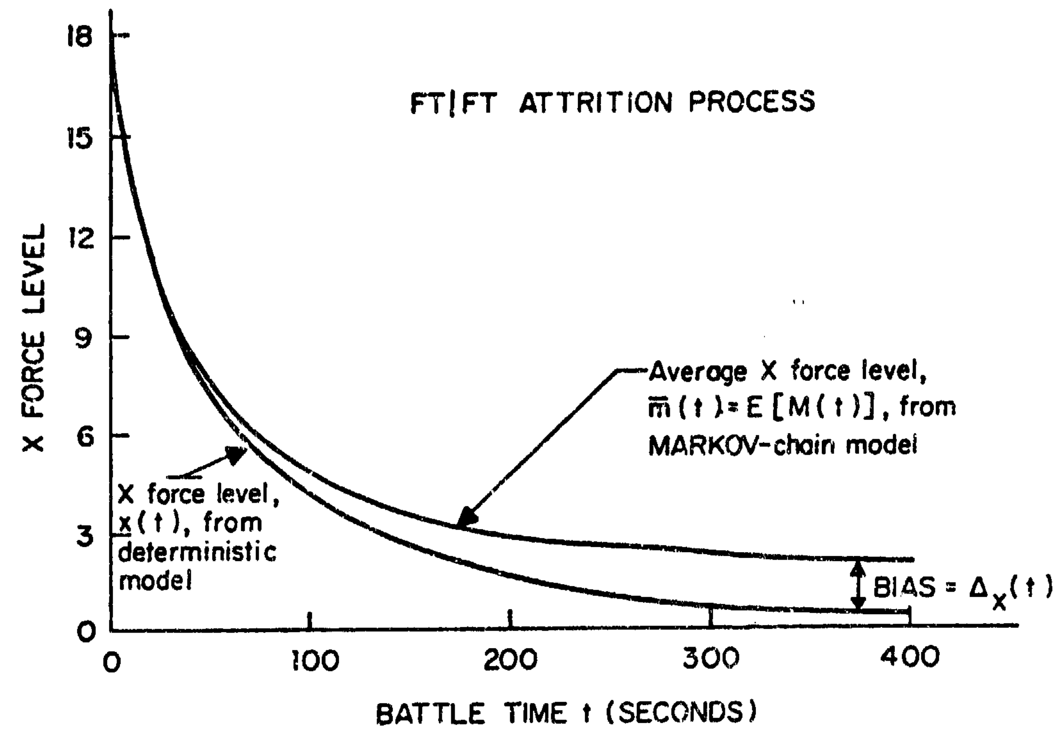


Figure 4.21. Bias $\Delta_X(t) = \bar{m}(t) - x(t)$ in the average X force level that is typical for small numbers of combatants in an FT|FT LANCHESTER-type stochastic attrition process. The input data for this case is $a = 0.004$ X (casualties/second) per Y firer, $b = 0.001$ Y (casualties/second) per X firer, $m_0 = 18$, and $n_0 = 6$.

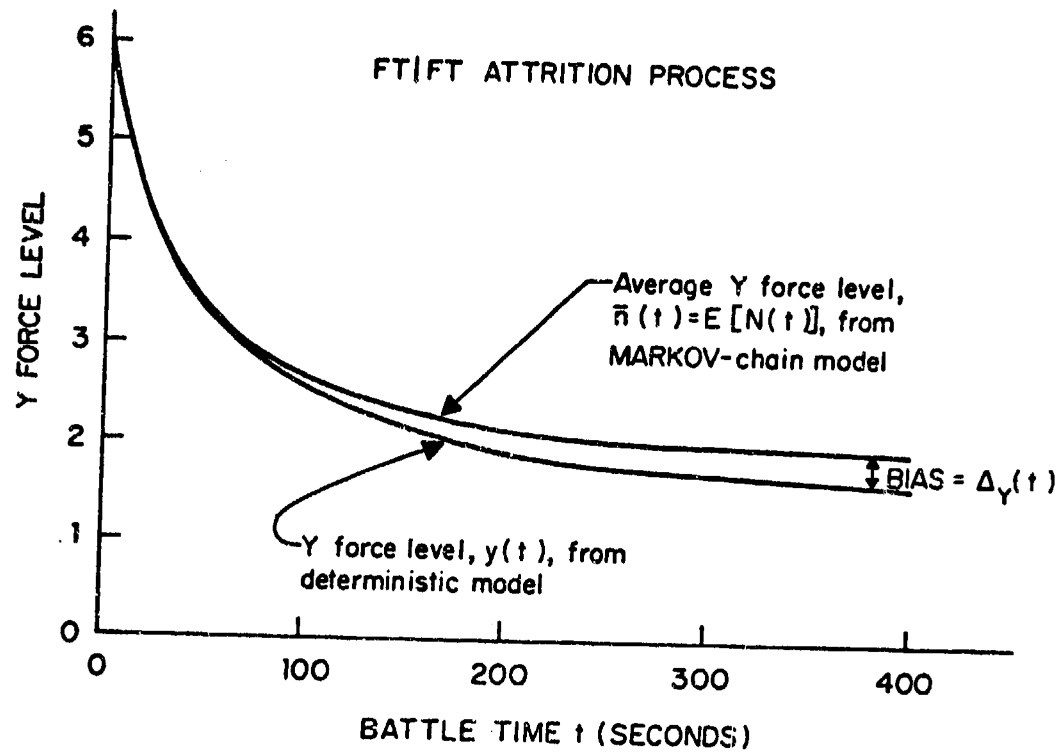


Figure 4.22. Bias $\Delta_Y(t) = \bar{n}(t) - y(t)$ in the average Y force level that is typical for small numbers of combatants in an FT|FT LANCHESTER-type stochastic attrition process. The input data for this case is $a = 0.004$ X (casualties/second) per Y firer, $b = 0.001$ Y (casualties/second) per X firer, $m_0 = 18$, and $n_0 = 6$.

4.13. Variability in the Mean Course of Combat.

Besides the mean course of combat itself, one is also interested in its variability (i.e. the dispersion of the numbers of survivors about their mean values) in order to gauge the risk in using these mean values to represent the probabilistic evolution of combat. Thus, we will now consider the variance and standard deviation of a combatant's force level. The reader will, of course, recall that the variance in, for example, X's force level $V_X(t)$ is given by

$$V_X(t) = E[M^2(t)] - \{E([M(t)]\}^2 , \qquad (4.13.1)$$

and the standard deviation is the square root of this quatity. Similar to as we saw in general for the calculation of the force-level moments in the previous section, there are essentially two methods for computing the variance of each side's force level. We will analogously refer to these two variance-calculation methods simply as follows:

(VM1) direct-computation method,

and (VM2) variance-covariance-differential-equation m^thod.

The reader should also recall from above in Section 4.9 that there are basically three methods of numerically calculating the joint probability distribution of the numbers of survivors for use in the variance-direct-computation method (VM1): the analytical method (M1), the numerical method (M2), and the hybrid analytical-numerical method (M3).

Little that was not said in the previous section about the moment-calculation methods remains to be said about the above variance-calculation methods. The variance-direct-computation method (VM1) certainly deserves no further discussion, and we will close this section with an example of the variance-covariance-differential-equation method (VM2). Let us first, however, review what various authors have found out and said about the variability in the mean course of combat.

Using analytical results for the distribution of survivors, F. C. BROOKS [13] concluded that the FT|FT and F|F stochastic LANCHESTER-type attrition processes (the latter only for the special case in which $a = b$) are stochastically determined. Here, <u>stochastically determined</u> means that the standard deviation in the losses is small compared to the initial numbers of weapons engaged. Following BROOKS [13, p. 2], we may then say that the model, although stochastic in detail, is very nearly deterministic in its gross behavior. BROOKS [13, p. 2] has stressed that "the presence of stochastic determinism suggests that the complex stochastic model may be subject to at least a crude approximation by a simpler deterministic model." On the other hand, G. CLARK [16] concluded that variability in the mean course of battle can be appreciable in small unit engagements, although his evidence for large battles (i.e. 100,000 or more combatants on each side) was not inconsistent with BROOKS' [13] conclusions on stochastic determinism.

Using his hybrid-analytical-numerical-computation method, CLARK [16, pp. 124-129] has computed the force-level variances for quite a few "typical" homogeneous-force battles in which attrition was modelled as F|F and FT|FT LANCHESTER-type stochastic processes. He concluded that the survivor

standard deviation depends on the following factors:

(F1) force size,

(F2) force ratio,

(FE) battle time,

and (F4) both attrition-rate coefficients.

Unfortunately, one does not know a priori what this dependence is. For battles between small numbers of combatants (i.e. under 15 on each side), CLARK observed that the magnitude of stochastic variability can be sizeable: the standard deviation approaches an asymptotic limiting value sometimes greater than one third of the initial force size and usually in the neighborhood of 15 percent.

Based on his computational studies for the F|F and FT|FT LANCHESTER-type stochastic attrition processes, CLARK [16, pp. 124-128] has hypothesized that there are two characteristic types of behavior for the survivor standard deviation of a force as a function of time:

(T1) the survivor standard deviation is an increasing function of time until a maximum value is achieved, and then it decreases to an asymptotic limiting value;

or (T2) the survivor standard deviation is an increasing function of time and is asymptotic to a limiting value.

Figures 4.23 and 4.24 show these two different types of behavior for the survivor standard deviation. Further computational studies on the F|F attrition process at the Naval Postgraduate School led CRAIG [19, p. 132] to conclude that the nature of the survivor standard deviation's time history is dependent on the relative attrition of the two opposing sides (i.e. the outcome of the battle). He has hypothesized that if a side has a high probability of winning, the standard deviation in its force level continually grows over time [i.e. type (T2) behavior occurs]. Furthermore, if a side has a high probability of losing, the standard deviation "peaks out" and then decreases to a asymptotic limiting value [i.e. type (T1) behavior occurs] (*see* CRAIG [19; pp. 127-134] for further details).

We will now close this section by developing the variance-covariance differential equation for the F|F LANCHESTER-type stochastic attrition process. Our results show that the variance-covariance-differential-equation method (VM2) fails to explicitly yield exact values for the sought quantities (i.e. the variances and covariance of the force levels) because one still needs to know $P(t,m,n)$ to be able to solve the system of differential equations. All is not lost, however, since some valuable insights into the variability of the mean course of combat are still obtainable. Thus, from the definitions

$$\bar{m} = E[M], \qquad \bar{n} = E[N]$$

$$\begin{aligned} \sigma_{XX} &= E[M^2] - E^2[M], \\ \sigma_{XY} &= E[MN] - E[M]\,E[N], \\ \sigma_{YY} &= E[N^2] - E^2[N], \end{aligned} \tag{4.13.2}$$

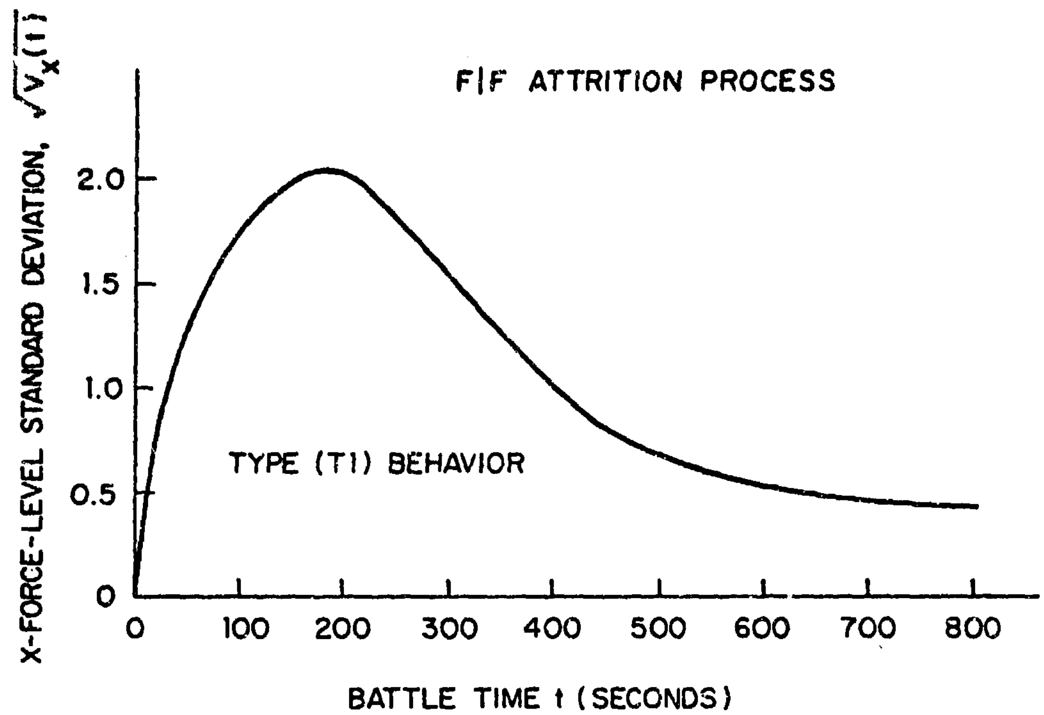


Figure 4.23. Type (T1) behavior for the survivor standard deviation. Shown here is the standard deviation of the X force level $\sqrt{V_X(t)}$ in an F|F LANCHESTER-type stochastic attrition process for the input data a = 0.004 X (casualties/second) per Y firer, b = 0.001 Y (casualties/second) per X firer, and $m_0 = n_0 = 8$.

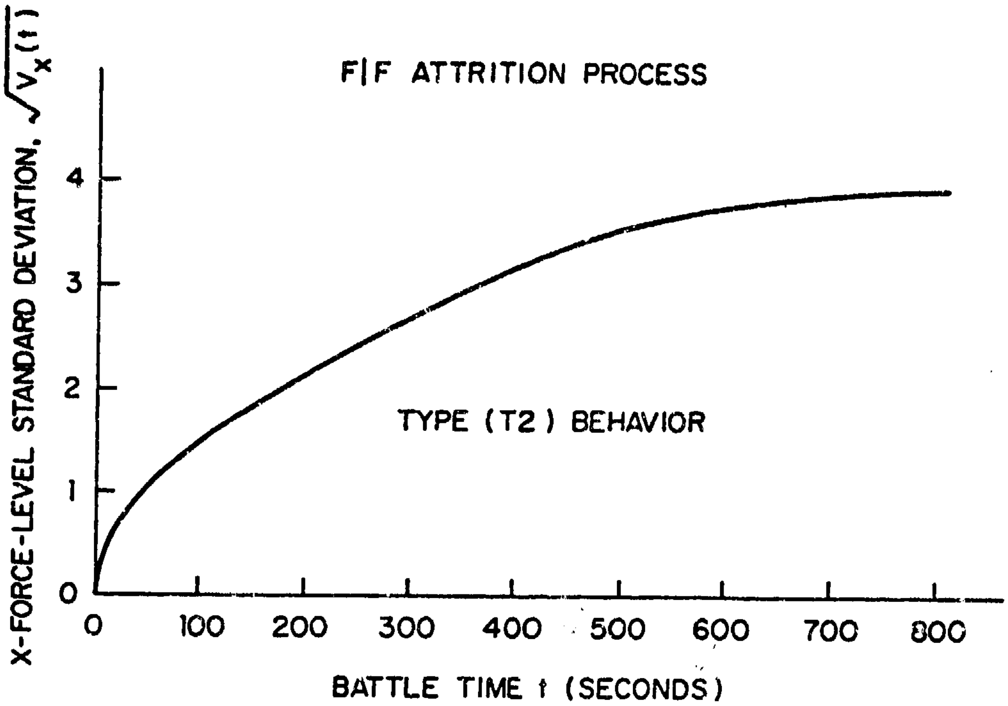


Figure 4.24. Type (T2) behavior for the survivor standard deviation. Shown here is the standard deviation of the X force level $\sqrt{V_X(t)}$ in an F|F LANCHESTER-type stochastic attrition process for the input data $a = 0.004$ X (casualties/second) per Y firer, $b = 0.0015$ Y (casualties/second) per X firer, $m_0 = 12$, and $n_0 = 6$.

and the relation (4.12.10), we find that for the F|F LANCHESTER-type stochastic attrition process

$$\begin{cases} \dfrac{d\bar{m}}{dt} = -a\bar{n} + aS_Y(t) & \text{with } \bar{m}(0) = m_0 , \\ \dfrac{d\bar{n}}{dt} = -b\bar{m} + bS_X(t) & \text{with } \bar{n}(0) = n_0 , \\ \dfrac{d\sigma_{XX}}{dt} = -2a\sigma_{XY} + a\bar{n} + S_{XX}(t) & \text{with } \sigma_{XX}(0) = 0 , \\ \dfrac{d\sigma_{XY}}{dt} = -b\sigma_{XX} - a\sigma_{YY} + S_{XY}(t) & \text{with } \sigma_{XY}(0) = 0 , \\ \dfrac{d\sigma_{YY}}{dt} = -2b\sigma_{XY} + b\bar{m} + S_{YY}(t) & \text{with } \sigma_{YY}(0) = 0 , \end{cases} \tag{4.13.3}$$

where $S_X(t)$ is given by (4.12.12), $S_Y(t)$ is given by (4.12.13),

$$\begin{aligned} S_{XX}(t) = 2a &\left\{ n_{BP} \sum_{m=m_{BP}+1}^{m_0} mP(t,m,n_{BP}) + m_{BP} \sum_{n=n_{BP}+1}^{n_0} nP(t,m_{BP},n) \right\} \\ &- a(2\bar{m} + 1) \left\{ n_{BP} \sum_{m=m_{BP}+1}^{m_0} P(t,m,n_{BP}) + \sum_{n=m_{BP}+1}^{n_0} nP(t,m_{BP},n) , \right\} \end{aligned} \tag{4.13.4}$$

$S_{XY}(t)$

$$= b\left\{\sum_{m=m_{BP}+1}^{m_0} m^2 P(tm,,n_{BP}) + m_{BP}^2 \sum_{n=n_{BP}+1}^{n_0} P(t,m_{BP},n)\right.$$

$$\left. - \bar{m}\left[\sum_{m=m_{BP}+1}^{m_0} mP(t,m,n_{BP}) + m_{BP} \sum_{n=n_{BP}+1}^{n_0} P(t,m_{BP},n)\right]\right\}$$

$$+ a\left\{n_{BP}^2 \sum_{m=m_{BP}+1}^{m_0} P(t,m,n_{BP}) + \sum_{n=n_{BP}+1}^{n_0} n^2 P(t,m_{BP},n)\right.$$

$$\left. - \bar{n}\left[n_{BP} \sum_{m=m_{BP}+1}^{m_0} P(t,m,n_{BP}) + \sum_{n=n_{BP}+1}^{n_0} nP(tm,m_{BP},n)\right]\right\} \text{ , (4.13.5)}$$

and $S_{YY}(t)$ is symmetric to $S_{XX}(t)$. It is unfortunately impossible to solve the above system of equations (4.13.3) without knowing $P(t,m,n)$ on the boundary of the state space.

In light of the above essentially insuperable difficulty, one is consequently quite tempted to approximate the solution to this system of differential equations by assuming that these latter state probabilities are negligible and solving the resultant simplified system. Thus, for $S_X(t) \equiv S_Y(t) \equiv S_{XX}(t) \equiv S_{XY}(t) \equiv S_{YY}(t) \equiv 0$, the above system (4.13.13) becomes

$$
\begin{cases}
\dfrac{d\hat{\bar{m}}}{dt} = -a\hat{\bar{n}} & \text{with} \quad \hat{\bar{m}}(0) = m_0 , \\[2ex]
\dfrac{d\hat{\bar{n}}}{dt} = -b\hat{\bar{m}} & \text{with} \quad \hat{\bar{n}}(0) = n_0 , \\[2ex]
\dfrac{d\hat{\sigma}_{XX}}{dt} = -2a\hat{\sigma}_{XY} + a\hat{\bar{n}} & \text{with} \quad \hat{\sigma}_{XX}(0) = 0 , \\[2ex]
\dfrac{d\hat{\sigma}_{XY}}{dt} = -b\hat{\sigma}_{XX} - a\hat{\sigma}_{YY} & \text{with} \quad \hat{\sigma}_{XY}(0) = 0 , \\[2ex]
\dfrac{d\hat{\sigma}_{YY}}{dt} = -ab\hat{\sigma}_{XY} + b\hat{\bar{m}} & \text{with} \quad \hat{\sigma}_{YY}(0) = 0 ,
\end{cases}
\tag{4.13.6}
$$

where $\hat{\bar{m}}$ denotes an approximation to $\bar{m}$, etc. An equivalent form of these equations was first given by SNOW [76, p. 25] in 1948. CLARK [16, pp. 130-132] has solved an equivalent form of this system of equations to find that the approximate variance in X's force level $\hat{V}_X(t) = \hat{\sigma}_{XX}(t)$ is given by

$$
\begin{aligned}
\hat{V}_X(t) = \frac{m_0}{2} - \frac{n_0}{2}\left(\frac{a}{b}\right) - \frac{(m_0 + n_0\, a/b)}{6}\cosh(2\sqrt{ab}\, t) + \left(\frac{m_0 + n_0}{3}\right)\sqrt{\frac{a}{b}}\, \sinh(2\sqrt{ab}\, t) \\
- \frac{(m_0 - 2n_0 a/b)}{3}\cosh(\sqrt{ab}\, t) - \left(\frac{2m_0 - n_0}{3}\right)\sqrt{\frac{a}{b}}\, \sinh(\sqrt{ab}\, t) ,
\end{aligned}
\tag{4.13.7}
$$

and similarly for $\hat{V}_Y(t)$. Unfortunately, this expression (4.13.7) is complicated enough that insights into how even the approximate force-level variance evolves over time are quite difficult (if not impossible) to obtain without expenditure of considerable computational effort. G. CLARK [16, pp. 133-134] has briefly numerically investigated the behavior of these expressions for the approximate force-level variances.

4.14. Monte Carlo Methods.

One can use so-called Monte Carlo methods to generate a realization of force-on-force combat modelled as a continuous-parameter MARKOV chain (or, for that matter, as any arbitrary but well-defined stochastic process) and hence to generate by appropriate replication battle data from which battle-summary statistics can be computed. Here we use the term Monte Carlo method to denote any procedure that utilizes statistical sampling techniques, involving the use of random numbers (more precisely, pseudorandom numbers [26, p. 171]), to determine the outcomes of random events. Although we will examine such methods here only within the context of developing estimates of desired statistics for simple LANCHESTER-type battles analytically modelled as continuous-time MARKOV chains, these methods can well be used with far more detailed models (i.e. ones enriched in operational details) that may not have such a corresponding simple analytical formulation. The reader should bear in mind that we will illustrate the basic ideas behind these Monte Carlo methods for analytical models for which other computational procedures (e.g. see the computational methods discussed in Section 4.9 above) are more efficient. Moreover, it is the author's firm opinion that many military OR analysts and (to be certain) military decision makers feel much more comfortable about using Monte Carlo methods because of their inherent concreteness (i.e. the generation of battle realizations) than they do about using analytical models directly, even when the desired information about system performance may be far more conveniently extracted from the appropriate corresponding analytical model (see BONDER [11, pp. 74-75] for further discussion).

This section is organized in the following fashion. First, we will discuss the simple analytical stochastic combat model to which we will

apply these methods and the analytical structures that will be utilized. Then, we will discuss in general terms how to generate needed samples of a given random variable from its cumulative distribution function and how to determine the outcome of a needed associated random event. Next, we will show how this general methodology is applied to the simple stochastic combat model under consideration. Finally, we will present an alternative Monte Carlo approach and will summarize the two approaches that could be used for this particular example.

We will consider the class of stochastic battles considered in Section 4.7, i.e. battles in which the attrition rates depend only on the combatants' force levels and not explicitly on time and which are modelled as continuous-time MARKOV chains. For such battles with stationary transition probabilities (4.7.1), we can build a simple Monte Carlo simulation based on the following two mathematical properties given in Section 4.7:

(P1) <u>the time between casualties</u> $T_{BC}^{m,n}$ <u>is exponentially distributed</u> (with state-dependent attrition rates), i.e.

$$P[T_{BC}^{m,n} \leq t] = 1 - \exp[-\{A(m,n) + B(m,n)\}t] \; ; \tag{4.14.1}$$

and (P2) <u>the probability that the next casualty will be an</u> X casualty is given by

$$P[X \text{ casualty} | \text{casualty occurs}] = \frac{A(m,n)}{A(m,n) + B(m,n)} \; . \tag{4.14.2}$$

If we are in battle state (m,n), then we will denote the next casualty to occur as the $i^{\underline{th}}$, where $i = m_0 - m + n_0 - n + 1$. We will denote the corresponding realization of $T_{BC}^{m,n}$ (<u>see</u> Section 4.6 for explanation of notation) simply as t_i. The total elapsed (or cumulative) time for the occurrence of the k^{th} casualty will therefore have realization t_k^{cum}, where

$$t_k^{cum} = \sum_{i=1}^{k} t_i \ . \tag{4.14.3}$$

The above two porperties (P1) and (P2) are used with a random-number generator that produces samples of a random variable that is uniformly distributed over the interval $[0, 1]$. Two samples from such a unit uniform variate are required to produce the realization of the occurrence of each battle casualty. In order to develop an algorithm to generate such realizations, we must discuss how the following two tasks are done by Monte Carlo methods:

(T1) generate a sample of a random variable (r.v.) for which the cumulative distribution function (c.d.f) is given,

and (T2) generate a sample of a binomial (or, more generally, multinominal) random variable that models the various possible outcomes of a random event with given probabilities of occurrence.

We therefore now turn to the first task (T1), to generate a sample of a r.v. with given c.d.f. Consider the continuously distributed r.v. V with distribution function $F_V(v) = P[V \leq v]$. Then for $u \in [0, 1]$, we may write

$$u = F_V(v) \ . \tag{4.14.4}$$

However, we may consider u and v to be realizations of the random variables U and V, with $U \in [0, 1]$ with certainty. Hence, we may write

$$U = F_V(V) \ , \tag{4.14.5}$$

whence

$$V = F_V^{-1}(U) \ , \tag{4.14.6}$$

where F_V^{-1} denotes the inverse function of the distribution function $F_V(u)$. Such an inverse function is well defined from the properties of the distribution function for a continuously distributed random variable. It may be shown (e.g. see FISHMAN [26, pp. 167-168]) that U defined by (4.14.5) is a uniformly distributed r.v. on $[0, 1]$. Thus, we can use (4.14.6) to generate samples of the r.v. V from samples of the unit uniform variate[28] U (see FISHMAN [26, p. 167]). This approach is, not unsurprisingly, known as the inverse-transformation method.

We may formalize the above approach by delineating the following procedure (composed of two steps) for generating a sample of the random

variable V with cumulative distribution function $F_V(v) = P[V \leq v]$ by the inverse-transformation method (see Figure 4.25, which is for the important special case in which $V \geq 0$):

(S1) generate sample of random variable U uniformly distributed on $[0, 1]$, call this u;

(S2) determine v such that $F_V(v) = u$, i.e. the desired sample value is $v = F_V^{-1}(u)$.

As shown in Figure 4.25, $P[V \leq v_0] = P[U \leq u_0]$ so that v_0 (generated by $v_0 = F_V^{-1}(u_0)$, where u_0 is a sample of a unit uniform variate) has all the statistical properties as a direct sample of the random variable V.

We may also use independent samples of the unit uniform variate U to generate realizations of discrete random events. In the simplest case, we consider the BERNOULLI random variable W, which takes on the value 1 with probability p and the value 0 with probability $(1-p)$. Thus, if we define

$$W = \begin{cases} 1 & \text{for } 0 \leq U \leq p , \\ 0 & \text{for } p < U \leq 1 , \end{cases} \tag{4.14.7}$$

then we can use samples of U to simulate sampling of W, since $P[0 \leq U \leq p] = P[W = 1]$. The above procedure is for a binominal r.v., and extension to a multinomial r.v. is carried out in the obvious manner.

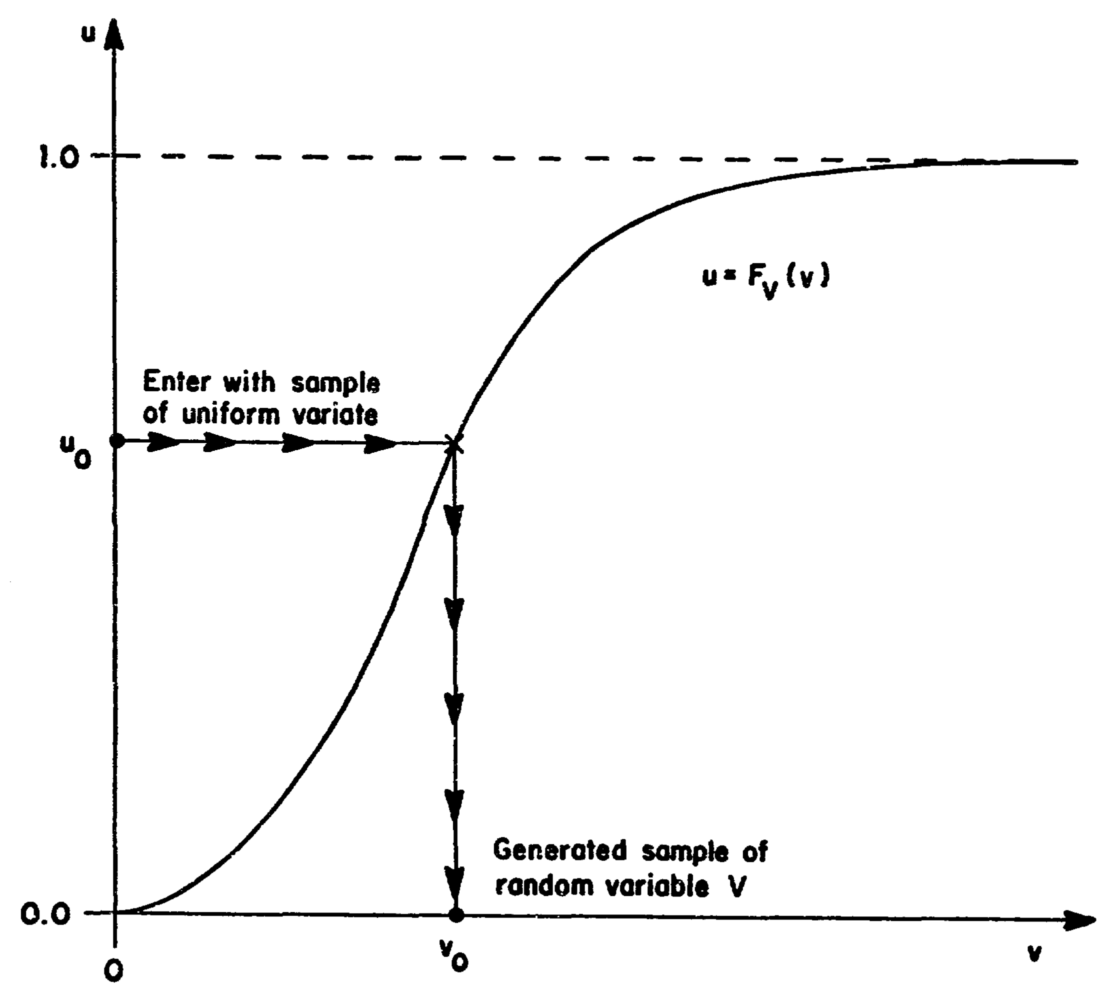


Figure 4.25. Basic idea of the inverse-transformation method for generating a sample of the random variable V. As discussed in the text, this approach utilizes the cumulative distribution function $F_V(v)$ of the random variable V (shown here for the important special case in which the random variable V is nonnegative).

One can now easily build a Monte Carlo simulation of such a stochastic attrition process based on the flow chart shown in Figure 4.26. To generate a realization of the random occurrence of a casualty in our continuous-time MARKOV-chain combat-attrition model, we must generate two independent samples of a unit uniform variate, denoted as u_1 and u_2, and operate on them in the following fashion. First, we use the first uniform-variate sample u_1 to generate a realization of the time of occurrence for the $i^{\underline{th}}$ casualty after the $(i-1)^{\underline{st}}$ one with the following formula

$$t_i = \frac{1}{\{A(m,n) + B(m,n)\}} \ \ell n\left(\frac{1}{1 - u_1}\right) , \tag{4.14.8}$$

which is just the inverse-transformation method (4.14.6) applied to the exponentially-distributed time between casualties (4.14.1). The time of occurrence of the $k^{\underline{th}}$ casualty t_k^{cum} is then given by (4.14.3). Next, we use the second uniform-variate sample u_2 to determine the type of casualty by the procedure that we have outlined above for binomial variates. Accordingly, we assess an X casualty if $0 \leq u_2 \leq A(m,n)/\{A(m,n) + B(m,n)\}$, and a Y casualty otherwise. Thus, one can build a Monte Carlo simulation to generate realizations of the stochastic homogeneous-force battle with stationary transition probabilities of Section 4.7, and extension to heterogeneous-force combat occurs in the obvious fashion (<u>see</u> CLARK [16, pp. 166-173] or ANDRIGHETTI [4, pp. 29-30] for further details).

We will close this section by briefly mentioning an alternative method (still involving, however, the generation of two independent unit-uniform-variate samples for each casulaty realization) for building such a

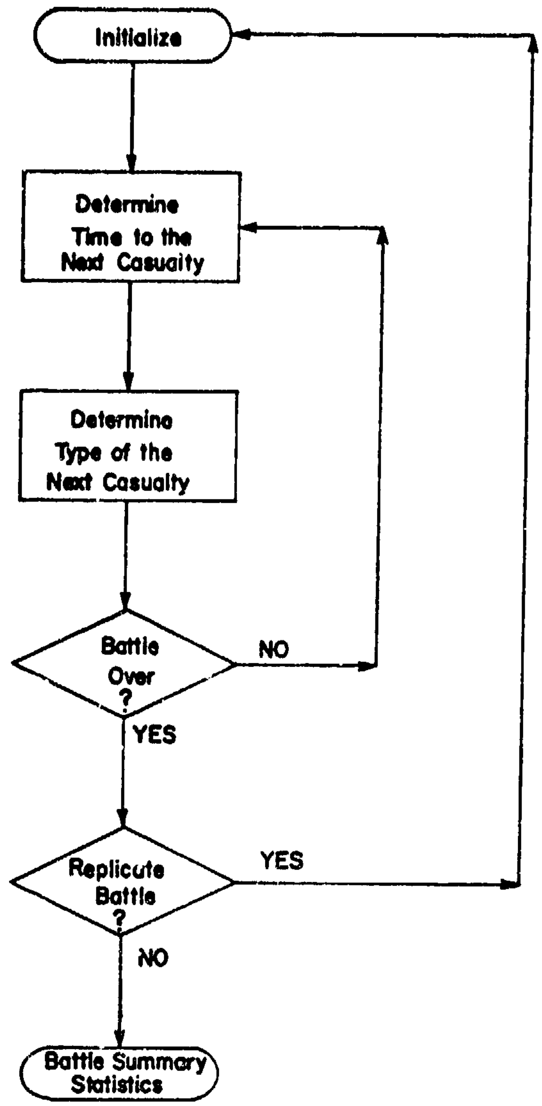


Figure 4.26. Flow chart of Monte Carlo simulation of continuous-time MARKOV-chain model of LANCHESTER-type attrition process. Two independent samples of a unit uniform variate are required for the realization of each casualty. The first is used to determine the time to the next casualty by (4.14.8) and the second to determine its type using (4.7.14).

Monte Carlo homogeneous-force simulation. Instead of basing our Monte Carlo approach on the two mathematical properties (P1) and (P2) given above, we could base it on the following two equivalent ones (again, cf. Section 4.7):

(P1') the time between occurrences of two X casualties $T_X^{m,n}$ is exponentially distributed with rate $A(m,n)$, i.e.

$$P[T_X^{m,n} \leq t] = 1 - \exp[-A(m,n)t]; \tag{4.14.9}$$

and (P2') the time between occurrences of two Y casualties $T_Y^{m,n}$ is exponentially distributed with rate $B(m,n)$, i.e.

$$P[T_Y^{m,n} \leq t] = 1 - \exp[-B(m,n)t]. \tag{4.14.10}$$

From these two properties (P1') and (P2'), one can develop a Monte Carlo method which generates a realization of the time interval to the next occurrence of both an X casualty and also a Y one and then takes the earlier of the two realizations to have occurred. Thus, we have outlined two equivalent Monte Carlo approaches:

Method A: generate time to next casualty and then determine type,

and Method B: generate time to next casualty of each type and then take earliest event to have occurred.

*4.15. Behavior for Large Numbers.

The results concerning approximations to the probability of winning that we have given in Section 4.11 above suggest that a limiting distribution for large numbers of combatants in the sense of the classic Central Limit Theorem of probability theory (e.g. see FELLER [25, p. 229]) lurks somewhere in the background of our stochastic combat model. In this section we will briefly consider a few heuristic arguments to shed some light onto this matter, particularly as they pertain to the mean course of combat. A complete mathematical discussion, however, would contain a number of subtle mathematical points that are well beyond the scope of our current cursory examination (see PERLA and LEHOCZKY [72] for details of such a deeper investigation invoking results from the theory of stochastic differential equations [5; 29] and diffusion approximations [24; 39]).

We begin by considering a heuristic argument that has appeared in a number of places in LANCHESTER combat theory (see WILLARD [91], ETTER [20], and KOOPMAN [55]). This argument will show us that as in so many other places in mathematical analysis, the taking of a limit can involve some subtle points. As pertains to MARKOVIAN combat-attrition processes, it means that transition from a discrete state space (i.e. MARKOV chain) to a continuous one (i.e. diffusion process) must be done in such a way that statistical properties of the process (i.e. means and variances of the force levels) are preserved.

Thus, we will (for illustrative purposes) consider the forward KOLMOGOROV equations for the F|F attrition process and heuristically investigate what happens as the number of each type of combatant becomes

large. Let us therefore rewrite (4.7.22) as

$$\frac{d}{dt} P(t,m,n) = an\{P(t,m+1,n) - P(t,m,n)\} + bm\{P(t,m,n+1) - p(t,m,n)\}. \qquad (4.15.1)$$

Strictly speaking, m and n can take on only nonnegative integer values. However, it is intuitively appealing to relax this restriction for large numbers of combatants and to replace m and n by x and y which are no longer restricted to be integers. Also, let us observe that when m was restricted to integer values, we could have written

$$m + 1 = m + \Delta m ,$$

i.e., $\Delta m = 1$, and similarly for Δn. Thus, we could rewrite (4.15.1) as

$$\frac{\partial}{\partial t} P(t,x,y) = ay \left\{ \frac{P(t, x + \Delta x, y) - P(t,x,y)}{\Delta x} \right\} + bx \left\{ \frac{P(t,x,y + \Delta y) - P(t,x,y)}{\Delta y} \right\} . \qquad (4.15.2)$$

For large numbers of combatants, Δx and Δy will be small compared to x and y, and consequently passing to the limit as $\Delta x \to 0$, $\Delta y \to 0$, and $P(t,x,y) \to p(t,x,y)$, we (naively) obtain the following first order partial differential equation (P.D.E.)

$$ay \frac{\partial p}{\partial x} + bx \frac{\partial p}{\partial y} - \frac{\partial p}{\partial t} = 0 , \qquad (4.15.3)$$

with initial condition

$$P(0,x,y) = \delta(x - x_0) \cdot \delta(y - y_0) , \qquad (4.15.4)$$

where $\delta(x)$ denotes the so-called DIRAC delta function which may be defined by[29] $\int_{-\infty}^{\infty} \phi(x)\, \delta(x - a)dx = \phi(a)$ for all $\phi(x) \in$ {appropriately defined class of "test" functions}. Here $p(t,x,y)$ denotes the joint probability density function for the X and Y forces, i.e.

$$p(t,x,y)dx\, dy = P\left[\begin{array}{l}\text{X force level between } x \text{ and } x + dx \text{ and Y} \\ \text{force level between } y \text{ and } y + dy \text{ at time } t.\end{array}\right]$$

Thus, our investigation of the behavior of our MARKOV-chain model of combat attrition has led us to the above first-order quasi-linear partial differential equation (P.D.E.) (4.15.3). To solve this equation for the joint probability density function $p(t,x,y)$, we will invoke the following result (for a proof and further details, see COURANT and HILBERT [18, pp. 62-69] or GARABEDIAN [28, pp. 18-22]; also HILDEBRAND [36, pp. 368-378]).

THEOREM 4.15.1 (MONGE): The solution to the first-order quasi-linear P.D.E. in the unknown function $z = z(x,y)$

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$$

is given by two independent integrals to the following system of first-order ordinary differential equations

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} \quad .$$

Thus, solving (4.15.3) by the method of characteristics (i.e. invoking Theorem 4.15.1), we find that

$$\frac{dx}{ay} = \frac{dy}{bx} = \frac{dt}{-1} = \frac{dp}{0} ,$$

or $p(t,x,y) = \text{CONSTANT}$ for

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with } x(0) = x_0 , \\ \dfrac{dy}{dt} = -bx & \text{with } y(0) = y_0 . \end{cases} \tag{4.15.5}$$

It is also easily shown using (4.15.4) and the definitions of average force levels, e.g. $\bar{x}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xp(t,x,y)dx\,dy$, that

$$\begin{cases} \dfrac{d\bar{x}}{dt} = -a\bar{y} & \text{with } \bar{x}(0) = x_0 , \\ \dfrac{d\bar{y}}{dt} = -b\bar{x} & \text{with } \bar{y}(0) = y_0 , \end{cases} \tag{4.15.6}$$

but that $V_X(t) \equiv V_Y(t) \equiv 0$, where (for example) $V_X(t)$ denotes the variance in X's force level. Thus, in passing to the limit for our stochastic combat model given by (4.7.19) through (4.7.24), we have recovered the deterministic battle equations for the F|F attrition process, but we have inadvertently destroyed the probabilistic nature of the model in a rather cavalier fashion. A more careful passage to the limit is required. A guiding principle in such a passage to the limit would be to preserve both the force-level means <u>and also variances</u>, which we have computed to be given by (4.13.3). Thus, our new results (4.13.3) should play an important role in developing approximations to solutions of the forward KOLMOGOROV equations.

A more careful passage-to-the-limit argument has been shown (see TAYLOR [78, pp. I-42 through I-46]) to yield the following diffusion approximation (e.g. see FELLER [24] or IGLEHART [39]) to the forward KOLMOGOROV equations

$$\frac{1}{2}\left\{a_1 y \frac{\partial^2 p}{\partial x^2} + b_1 x \frac{\partial^2 p}{\partial y^2}\right\} + ay \frac{\partial p}{\partial x} + bx \frac{\partial p}{\partial y} = \frac{\partial p}{\partial t}, \tag{4.15.7}$$

with initial condition (4.15.4). Here $a_1 > 0$ and $b_1 > 0$ insure that the probability density diffuses over time (cf. the figures in Section 4.9). Unfortunately, finding a solution to the parabolic P.D.E. (4.15.7) has proven to be quite elusive. However, PERLA and LEHOCZKY [72] have invoked results from the theory of stochastic differential equations (e.g. see GIHMAN and SKOROHOD [29] or ARNOLD [5]) to develop a diffusion approximation based on the stochastic differential equations corresponding to the parabolic P.D.E. (4.15.7). PERLA and LEHOCZKY [72] have thereby obtained the approximate force-level-mean-and-variance equations (4.13.6), which we have obtained as approximations to the exact system of equations (4.13.3) without any assumption concerning large numbers of combatants. Furthermore, our development in Section 4.13 shows that (4.13.6) is a good approximation to (4.13.3) only as long as a significant amount of probability has not accumulated on the boundary of the state space[30]. Thus, we feel that the exact force-level-mean-and-variance equations (4.13.3) developed in Section 4.13 should prove quite useful for developing approximations to the forward KOLMOGOROV equations in the future.

Finally, it should be pointed out that both G. CLARK [16, pp. 133-134] and also PERLA and LEHOCZKY [72] (the latter authors also giving results for other attrition processes) have numerically investigated the behavior of the approximate force-level-mean-and-variance equations for the F|F attrition process. CLARK [16, p. 137] concluded that the force-level variability for larger units (i.e. large numbers) is larger in absolute terms but is a much smaller percentage of the force size than for small units. PERLA and LEHOCZKY [72, p. 26] concluded that the approximate equations (4.13.6) provide "good results" for initial force levels as small as 30 on each side. Extensive numerical computations and representative results for the approximate force-level means and standard deviations and a comparison of these approximations with exact Monte Carlo results (cf. Section 4.14) have also been reported by these latter authors.

4.16. Comparison of Deterministic and Stochastic Attrition Models.

In this section we will consider the important question, "How do random fluctuations in the occurrence of casualties modify the results obtained from determinstic LANCHESTER-type combat models?" A number of authors (e.g. SPRINGALL [77], CLARK [16], and CRAIG [19]) have investigated various aspects of this very important question in considerable detail, and we will summarize their findings later in this section. However, there is a broader context in which we can view this question and which is more consistent with the research philosophy espoused several places elsewhere in this monograph (*cf*. Sections 4.4 and 6.3): *we can view a stochastic force-on-force combat model as an abstraction of reality that should capture the essence of the combat-attrition process and provide information on the essential underlying dynamics of combat*. This information itself should, of course, be responsive to the demands of military OR/systems analysis for defense-planning purposes. The essential underlying question concerning the comparison of deterministic and stochastic force-on-force attrition models is then, "How does the information that each combat-model type provides on the dynamics of combat compare and influence defense decision making?" Unfortunately, we will only be able to address the first aspect (i.e. comparison of information generated by each combat-model type) here, but the entire issue ultimately rests on the second.

Viewed within the context of information provided on the dynamics of combat, one single model can never be sufficient for the purposes of military OR. We should instead consult several different complementary models that provide information over a spectrum of issues at various levels of detail. Within this context, the crucial question concerning the comparison of

deterministic and stochastic force-on-force attrition models is whether they both provide consistent information about the dynamics of combat. If (for example) a deterministic force-on-force attrition model provides "first-order" information about trends, while a stochastic one provides "second-order" information about these trends that merely refines rather than revises, then the choice of the more appropriate type of model rests solely on how refined an answer is desired. This is the basic conclusion drawn by this author after considering many different aspects of the problem and the results of many authors. In other words, the deterministic models serve to provide us with a basic orientation about the dynamics of combat, and the stochastic models usually serve to refine this orientation. There are, of course, exceptions and future research should concentrate on more clearly delineating when such exceptions occur.

Basically, the deterministic models provide information much more conveniently about the dynamics of combat than the corresponding stochastic models do. In general terms, a stochastic combat model provides distributional information about combat outcomes. However, not only is such probabilistic information relatively difficult to extract from the stochastic model but one must also employ summary statistics to transform it into a less complicated and more convenient form for decision making. Moreover, <u>if such distributional information is not used, then we do not learn any more from the stochastic model than from the corresponding original deterministic model</u>. Along these lines, such advantages and disadvatnages of deterministic and stochastic LANCHESTER-type combat models are summarized in Table 4.IV.

It is the opinion of this author that deterministic LANCHESTER-type combat models <u>do</u> capture the "first-order" trends of combat dynamics[31] except

TABLE 4.IV. Advantages and Disadvantages of Deterministic and Stochastic LANCHESTER-Type Combat Models.

Deterministic LANCHESTER-Type Combat Models

ADVANTAGES	DISADVANTAGES
1. Information easily extracted from model 2. Dynamics of combat transparently revealed	1. Further abstraction from reality (randomness suppressed)

Stochastic LANCHESTER-Type Combat Models

ADVANTAGES	DISADVANTAGES
1. Closer to reality in sense that one type of randomness explicitly portrayed	1. Information not easily extracted from model (and only then with considerable computational cost) 2. Dynamics of combat (i.e. driving factors) not readily revealed

for small numbers of potential casualties (i.e. each side can take under 20 casualties before its breakpoint is reached) and initial conditions of near parity[32]. However, more experimental computation and theoretical work is required to identify more precisely a priori circumstances under which results from the deterministic models may be misleading. We will now review what other authors have concluded about this important subject.

Work on the topic of comparison of deterministic and stochastic LANCHESTER-type combat models falls naturally into three chronological categories:

(C1) that done before the work of SPRINGALL [77] and CLARK [16],

(C2) the work of SPRINGALL [77] and CLARK [16],

and (C3) that done after the work of SPRINGALL [77] and CLARK [16].

The work of SPRINGALL [77] and that of CLARK [16] must be regarded as definitive, were done simultaneously and independently of each other, and reached apparently contradictory conclusions (SPRINGALL [77, p. 186] concluding that a deterministic formulation captures the essence of trends in such combat dynamics and CLARK [16, p. 243] concluding that deterministic models are inadequate for small-unit engagements). Work before that of these two authors never systematically examined the issue of deterministic versus stochastic LANCHESTER-type combat models as deeply, and subsequent work has tended to take their two Ph.D. theses as a point of departure.

Work in the first chronological category (C1) was done far before LANCHESTER-type combat models had had any widespread application to defense-planning problems and also before the widespread routine use of the large-scale digital computer in general scientific computation. The classic book by MORSE and KIMBALL [65, pp. 63-71] contains not only the earliest work in the western world (exclusive of LANCHESTER's [56] original 1914 paper) on simple deterministic-type models but also the earliest work on stochastic ones and comparison of the deterministic and stochastic models [65, pp. 67-71]. MORE and KIMBALL made such comparisons for the F|F and the FT|FT LANCHESTER-type attrition processes[33] and concluded [65, p. 69 and p. 71] that "as long as the equations are not pressed too hard (such as by going to the annihilation of one force)" such deterministic models do capture the first-order probabilistic trends in the dynamics of combat and may be thought of as representing the mean course of combat (i.e. such deterministic models may be thought of as "expected-value" models). For both the F|F and FT|FT attrition processes, MORSE and KIMBALL have compared some expected values from the stochastic model with force levels obtained from the corresponding deterministic model. For example, for the stochastic F|F LANCHESTER-type attrition process, MORSE and KIMBALL [65, Tables 3 and 4 on p. 70] (see also SNOW [76, pp. 26-27]) explicitly solved the applicable system of 23 forward KOLMOGOROV equations in a special case (i.e. $m_0 = 5$, $n_0 = 3$, and $a = b = 1.0$) and calculated $\lim_{t \to +\infty} E[M(t)]$ and $\lim_{t \to +\infty} E[N(t)]$ to obtain the results shown in Table 4.V. SNOW [19] later examined such questions in greater depth[34]. He [76, p. 27] apparently first obtained the average-force-level equations for the F|F attrition process (4.12.11) in the special case of a fight to the

TABLE 4.V. Comparison of End-of-Battle Results from Deterministic and Stochastic Models for the F|F Attrition Process and a Fight to the Finish.

	Limiting Value of Expected Force Level from Stochastic Model	Force Level from Deterministic Model
X Force Level*	3.492	4.000
Y Force Level	0.232	0.000

*The number in the first column represents $\lim_{t \to +\infty} E[M(t)]$ as determined from the stochastic model, while that in the second represents $x(t_f)$ as determined from the corresponding deterministic model when $y(t_f) = 0$.

finish, i.e. $m_{BP} = n_{BP} = 0$. SNOW [76, p. 28] concluded that deterministic LANCHESTER-type equations (at least for the special case of the F|F attrition process) could be considered as approximations to the mean-value equations for the corresponding continuous-time MARKOV-chain model (cf. Section 4.12). This material on comparing deterministic and stochastic LANCHESTER-type combat models presented by MORSE and KIMBALL [65] and SNOW [76] in some sense represents the point of departure for G. CLARK's [16] much more detailed and comprehenisve investigation of this important modelling issue. Again, it should be emphasized that the need for such a theoretical comparison did not appear to be very important to military OR workers until circumstances (including the advent of the large-scale digital computer) had led to the widespread use of computer-based-combat-model decision aids by the U. S. Department of Defense in the later 1960's. At that time this theoretical modelling issue assumed practical significance for guiding the development of operational models.

Other work in the first chronological category (C1) has considered other aspects and issues subsequently investigated in the stochastic-versus-deterministic-models debate. SNOW [76, p. 25] apparently first raised the question about considering the higher moments (in particular, the variability) of the stochastic results. BROOKS [13, p. 1] suggested that survivor variability (more precisely, survivor standard deviation) generally decreases for many combat models in relation to the initial force levels as the latter are increased. He concluded that sochastic determinism (see [13, pp. 1-2] or Section 4.13 above) exists for many force-on-force combat models and that there is an approximately deterministic relation between the initial conditions

and the gross results. WILLARD [91] (see also KOOPMAN [55] and ETTER [20]) suggested that the stochastic F|F LANCHESTER-type attrition process converges to a deterministic process as force sizes increase (see Section 4.15 above, however), and CLARK [16, p. 54] subsequently perceived confusion on this point of force-level variability. Other points of comparison have been investigated by G. H. WEISS [89, 90], whose efforts more or less anticipated the later more detailed and comprehensive investigations of SPRINGALL [77]. G. H. WEISS [89; 90] concluded that the deterministic models produced results not contradicted by stochastic-model results and that such qualitative agreement increases as the numbers of combatants increases.

SPRINGALL [77] obtained time-dependent results (in particular, the distribution for the duration of battle) for all the simple LANCHESTER-type stochastic combat models and developed expressions for the victory probabilities and distribution of survivors for some more complicated models. He then used these results to investigate the issue of stochastic versus deterministic combat models. SPRINGALL [77, p. 152] pointed out that although deterministic and stochastic LANCHESTER-type models yield results that are (at first sight at least) entirely different in their basic nature (although based on equivalent premises), one would expect that the two types of models lead to approximately the same conclusions concerning (cf. the analysis questions presented in Table 4.I):

(Q1) Who will be the victor?

(Q2) How many survivors will he have?

(Q3) How long will the battle last?

SPRINGALL argued that the most important comparison criterion was the prediction of battle outcome and that if the two types of models could not agree on this, there would be little hope for agreement on subsidiary attributes. Based on both theoretical and also many numerical comparisons for a number of different models [77, pp. 151-166], SPRINGALL [77, p. 167] concluded that

(C1) the deterministic results most adequately describe force-on-force attrition in combat when the number of combatants on each side is large[35],

and (C2) although it is capable of providing reasonable approximations to the expected values of the force levels, a deterministic model (by its very nature) cannot give any insight into their variances (which can be appreciable).

These conclusions were based on consideration of numerical results for a variety of models concerning (1) prediction of battle outcome, (2) the average force levels, (3) force-level variability, and (4) the expected duration of battle. At the end of this thesis, SPRINGALL [77, p. 187] has reiterated his general conclusion concerning these two basic types of models that (where appropriate) a deterministic LANCHESTER-type combat model is increasingly valid as an approximation to the results from a corresponding stochastic (i.e. continuous-time MARKOV-chain) model as the number of combatants on each side increases. The stochastic results however, did not seem to converge completely to the deterministic results under all conditions. This latter point apparently did not concern SPRINGALL very much (see [77, p. 187] for

further details), although it was the source of much concern for CLARK [16].

G. M. CLARK [16] has presented an even more detailed and comprehensive comparison of deterministic and stochastic LANCHESTER-type combat models in his Ph.D. thesis. Much of his supporting analysis has already been presented in Sections 4.9, 4.12, and 4.13 above, and consequently complete details need not be given here. CLARK [16, p. 243] concluded that "analysis of the bias in LANCHESTER combat models and of the survivor standard deviation supports the choice of stochastic models over deterministic models when describing small unit engagements" (say under 12 weapons on each side). Here CLARK [16, p. 59] took the term <u>bias in the deterministic model</u> to mean the difference between the deterministic model's result and the expected state of the corresponding stochastic attrition process. CLARK [16, p. 243], however, added that for large-unit engagements (say, over 100,000 combatants on each side) results indicated that deterministic force-on-force combat models are (for the most part) satisfactory, i.e. survivor bias and stochastic variability appear negligible. CLARK apparently based these conclusions on rather extensive computations of force-level means and standard deviations for the F|F and FT|FT stochastic LANCHESTER-type combat models and comparison of these stochastic-model results with the appropriate corresponding deterministic-model results. Some of these computational results are shown in Tables 4.VI and 4.VII (<u>see</u> CLARK [16] for further details; also <u>see</u> the figures (extracted from his Ph.D. thesis) in Sections 4.12 and 4.13 above). It should be pointed out that from CLARK's computational results it appears that he took the term "small-unit engagement" to mean <u>approximately 12 or less combatants on each side</u>. Thus, SPRINGALL's and G. CLARK's conclusions about the comparative quantitative behavior of

TABLE 4.VI. Typical Biases $\Delta_X(t)$ and $\Delta_Y(t)$ in the Average Force Levels in an FT|FT LANCHESTER-type Stochastic Attrition Process and a Fight-to-the-Finish (from CLARK [16]).

Initial Force Levels		Effectiveness of Individual Firers		Battle Time t (seconds)	Average Force Levels		Deterministic Model		Bias in Average Force Level	
m_0	n_0	a (Y combatant)	b (X combatant)		$\bar{m}(t)$	$\bar{n}(t)$	$x(t)$	$y(t)$	$\Delta_X(t)$	$\Delta_Y(t)$
6	6	0.004	0.001	900.	0.024	4.506	0.000	4.500	0.024	0.006
6	6	0.004	0.0015	1075.	0.107	3.790	0.000	3.750	0.107	0.040
6	6	0.004	0.002	1275.	0.269	3.134	0.000	3.000	0.269	0.134
6	6	0.004	0.004	2500.	1.350	1.350	0.093	0.093	1.257	1.257
8	8	0.004	0.001	400.	0.020	6.005	0.000	6.000	0.020	0.005
8	8	0.004	0.0015	350.	0.107	5.040	0.005	5.002	0.102	0.038
8	8	0.004	0.002	1000.	0.214	4.107	0.000	4.000	0.214	0.107
8	8	0.004	0.004	2500.	1.571	1.571	0.099	0.099	1.472	1.472
12	6	0.004	0.001	850.	0.374	3.094	0.000	3.000	0.374	0.094
12	6	0.004	0.0015	1500.	1.221	1.958	0.000	1.500	1.221	0.458
12	6	0.004	0.002	2500.	2.351	1.175	0.197	0.098	2.154	1.077
12	6	0.004	0.004	375.	6.149	0.149	6.000	0.000	0.149	0.149

NOTE: The bias in the average X force level $\Delta_X(t)$ is defined as $\Delta_X(t) = \bar{m}(t) - x(t)$ and similarly for $\Delta_Y(t)$.

TABLE 4.VII. Typical Terminal Values for the Standard Deviation of a Combatant's Force Level in an FT|FT LANCHESTER-type Stochastic Attrition Process and a Fight-to-the-Finish (from CLARK [16]).

Initial Force Levels m_0	n_0	Effectiveness of Individual Firers a (Y combatant)	b (X combatant)	Battle Time t (seconds)	Standard Deviation of Force Level $\sqrt{V_X(t)}$	$\sqrt{V_Y(t)}$
6	6	0.004	0.001	1500.	0.241	1.346
6	6	0.004	0.0015	1500.	0.530	1.642
6	6	0.004	0.002	1500.	0.840	1.811
6	6	0.004	0.004	2500.	1.678	1.678
8	8	0.004	0.001	1500.	0.158	1.571
8	8	0.004	0.0015	1500.	0.441	1.954
8	8	0.004	0.002	1500.	0.808	2.192
8	8	0.004	0.004	2500.	1.990	1.990
12	6	0.004	0.001	1500.	1.283	1.723
12	6	0.004	0.0015	1500.	2.271	1.743
12	6	0.004	0.002	2500.	2.987	1.493
12	6	0.004	0.004	1500.	3.120	0.565

deterministic and stochastic LANCHESTER-type combat models are not really contradictory. Each researcher was apparently considering a different realm of applicability as regards initial force levels, CLARK ten or less on each side and SPRINGALL over thirty.

We next turn to work in the last chronological category (C3), which has appeared subsequent to that of SPRINGALL [77] and CLARK [16]. It is indeed paradoxical that today when LANCHESTER-type models are more widely used than ever before for defense planning in the U.S. DoD and elsewhere (many times for investigating very practical operational-planning questions), very little material is being published on such models, with next to nil about stochastic variations[36] (and with at least even an order of magnitude less information appearing about comparisons of stochastic with deterministic LANCHESTER-type combat models). To be sure, there are significant research activities going on, but for various reasons most of it does not ever get documented and/or published[37]. With this important qualification being observed, we will now briefly examine what investigations concerning comparison of deterministic and stochastic LANCHESTER-type combat models have appeared subsequent to 1969. The major pieces of work concerning such comparisons known to this author are by CRAIG [19] and KARR [47; 48]. The former author presents far more computational results than does the latter author, and consequently we will focus on CRAIG's work [19], which is corroborated by that of KARR [47; 48].

CRAIG [19] took SPRINGALL's [77] and CLARK's [16] work as a point of departure and did extensive numerical calculations upon which he based his comparison of deterministic and stochastic LANCHESTER-type combat models.

He considered only the F|F attrition process, though. CRAIG [19, p. 140] concluded that the complex random process of force-on-force combat can be adequately represented by a deterministic model if

(1) there are at least 20 combatants on each side,

(2) force-level breakpoints are such that each side is willing and capable of taking at least 20 casualties,

(3) the forces are not near parity,

and (4) if near parity, then each side initially has at least 40 combatants and is willing and capable of taking at least 20 casualties.

In other words, unless the two homogeneous forces are near parity, essentially the same information about the dynamics of combat is obtained from both deterministic and stochastic LANCHESTER-type combat models (i.e. the models are not significantly different in terms of the outputs that they produce, at least in qualitative terms). CRAIG [19] based these conclusions on extensive computations of the probability of winning and the force-level means and variances (<u>see</u> Tables 4.VIII and 4.IX for some representative computational results from [19]).

TABLE 4.VIII. Typical Biases $\Delta_X(t)$ and $\Delta_Y(t)$ in the Average Force Levels (Also Expressed as a Percent of Total Loss in Corresponding Deterministic Battle) in an F|F LANCHESTER-Type Stochastic Attrition Process and a Fixed-Force-Level-Breakpoint Battle (from CRAIG [19]).

Initial Force Levels m_0	Initial Force Levels n_0	Effectiveness of Individual Firers a (Y combatant)	Effectiveness of Individual Firers b (X combatant)	Breakpoint Parameters f_{BP}^X	Breakpoint Parameters f_{BP}^Y	Battle Time t (minutes)	Bias in Average Force Level $\Delta_X(t)$	Bias in Average Force Level $\Delta_Y(t)$	Bias as a Percent of Total Loss in Deterministic Battle for X Force	Bias as a Percent of Total Loss in Deterministic Battle for Y Force
12	12	0.008	0.004	0.00	0.00	155.81	1.75	-0.20	14.56	-5.70
24	24	0.008	0.004	0.00	0.00	155.81	2.48	-0.22	10.33	-3.13
40	40	0.008	0.004	0.00	0.00	155.81	3.22	-0.23	8.05	-1.96
24	24	0.008	0.004	0.20	0.20	120.68	2.10	0.06	10.50	0.68
40	40	0.008	0.004	0.20	0.20	120.68	2.62	0.18	8.19	1.61
24	24	0.008	0.004	0.40	0.40	86.86	1.68	0.29	11.20	4.93
40	40	0.008	0.004	0.40	0.40	86.86	2.11	0.42	8.79	4.40
24	24	0.008	0.004	0.60	0.60	55.25	1.31	0.38	13.10	8.74
40	40	0.008	0.004	0.60	0.60	55.25	1.64	0.52	10.25	7.41
24	24	0.008	0.004	0.80	0.80	26.29	0.94	0.38	18.80	16.10
40	40	0.008	0.004	0.80	0.80	26.29	1.15	0.47	14.38	12.43
40	40	0.004	0.0015	0.40	0.40	166.41	2.06	0.28	8.58	4.06
40	40	0.004	0.0015	0.60	0.60	107.47	1.64	0.52	10.25	7.41
40	40	0.004	0.0015	0.80	0.80	51.88	1.15	0.47	14.38	12.43

NOTES: 1. The bias in the average X force level $\Delta_X(t)$ is defined as $\Delta_X(t) = \bar{m}(t) - x(t)$ and similarly for $\Delta_Y(t)$.

2. The force-level breakpoint for the X force in the stochastic battle is given by $m_{BP} = [f_{BP}^X m_0]$, where $[x]$ denotes "the greatest integer in x," and similarly for n_{BP}.

TABLE 4.IX. Typical Terminal Values for the Standard Deviation of a Combatant's Force Level in an F|F LANCHESTER-Type Stochastic Attrition Process and a Fixed-Force-Level-Breakpoint Battle (from CRAIG [19]).

Initial Force Levels m_0	Initial Force Levels n_0	Effectiveness of Individual Firers a (Y combatant)	Effectiveness of Individual Firers b (X combatant)	Force-Level Breakpoints m_{BP}	Force-Level Breakpoints n_{BP}	Battle Time t (minutes)	Standard Deviation of Force Level $\sqrt{V_X(t)}$	Standard Deviation of Force Level $\sqrt{V_Y(t)}$
12	12	0.008	0.004	0	0	155.81	2.39	2.49
24	24	0.008	0.004	0	0	155.81	3.45	3.59
40	40	0.008	0.004	0	0	155.81	4.50	4.67
24	24	0.008	0.004	4	4	120.68	2.90	3.27
40	40	0.008	0.004	8	8	120.68	3.66	4.18
24	24	0.008	0.004	9	9	86.86	2.34	2.81
40	40	0.008	0.004	16	16	86.86	2.97	3.61
24	24	0.008	0.004	14	14	55.25	1.85	2.26
40	40	0.008	0.004	24	24	55.25	2.32	2.94
24	24	0.008	0.004	19	19	26.29	1.35	1.56
40	40	0.008	0.004	32	32	26.29	1.71	2.04
40	40	0.004	0.0015	16	16	166.41	2.89	2.94
40	40	0.004	0.0015	24	24	107.47	2.29	2.46
40	40	0.004	0.0015	32	32	51.88	1.68	1.81

Based on careful review of all the work discussed above, we have reached the conclusions regarding the relative merits of deterministic and stochastic LANCHESTER-type combat models that we have presented earlier in this section: in essence, <u>unless the force levels are appreciably below 20 on each side and the forces near parity, a deterministic LANCHESTER-type model is quite adequate for representing force-on-force attrition</u> (particularly if distributional information is not required). These conclusions must be regarded as somewhat tentative, however, and more computational and theoretical work is required to more clearly delineate when exceptions to the above general rule of thumb should be expected to occur.

FOOTNOTES FOR CHAPTER 4

1. The principal subsequent works on stochastic combat models that have appeared in the unclassified literature are the ones by SNOW [76], BROWN [14; 15], BROOKS [13], D. G. SMITH [75], SPRINGALL [77], CLARK [16; 17], GRUBBS and SHUFORD [33], and BOWEN [25]. The reader may also find it worthwhile to read the more recent paper by KOOPMAN [55]. It contains a number of interesting conceptual ideas about the representation of combat attrition as a stochastic process. For the sake of completeness we also note here the following papers: ISBELL and MARLOW [40], MARADUDIN and WEISS [59], WILLARD [91], C. MARSHALL [61], KISI and HIROSE [53], MARMA and DEUTSCH [60], MJELDE [64], JAIN and NAGABHUSHANAM [41], SHUFORD and GRUBBS [74], FARRELL and FREEDMAN [22], WATSON [81], GYE and LEWIS [34], and GOLDIE [31]. Finally, mention should be made of the eight reports from the Defence Operational Analysis Establishment (DOAE) (of the Ministry of Defence of the United Kingdom) by WEALE [82-86], JENNINGS [42-43], and WEALE and PERYER [87], the work at the Institute for Defense Analyses by KARR [45-49], and the M.Sc. thesis by GRAINGER [32]. The first seven DOAE reports [42-43; 82-87] have been reviewed and critiqued by KARR [50]. No further readily available work on stochastic LANCHESTER-type models could be found in the recently published list of references on the LANCHESTER theory of combat by HAYSMAN and MORTAGY [35].

2. BARTLETT [7] has pointed out that many different stochastic birth-and-death models are compatible with a given deterministic differential-equation population-growth model. Clearly, the same is true for combat attrition models. For example, MARADUDIN and WEISS [59] and G. H. WEISS [89] have considered different stochastic birth-and-death models that are compatible with the deterministic FT|FT attrition process.

3. Similar stochastic models arise in various fields of science and technology such as mathematical biology [7; 8; 30; 51; 58; 89], ecology [73], epidemiology [6; 90], etc. (e.g. see BHARUCHA-REID [10] for some further fields of application in which such similar models arise). Further references to the literature are to be found in the above cited work, which may be taken as an abbreviated (but selective) guide for further reading. In particular, a fairly extensive literature on stochastic population models exists and is readily available to the interested reader (see KENDALL [51] for a review and summary of earlier, i.e. pre-1950, work on stochastic population models, while more recent activities have been reported in BARTLETT [8], GOEL et al. [30], and LUDWIG [58]). It should be pointed out, however, that although concepts and even most details of model formulation are essentially the same in these related fields and LANCHESTER combat theory, few results from these allied fields have been found to be directly applicable to LANCHESTER-type combat models (see G. H. WEISS [89] for a notable exception, though). The reasons for this state of affairs are apparently that (R1) somewhat different information about a model is required in these related fields (cf. Table 4.I above),

and (R2) combat models (in contrast to such related models) apply to systems with bounded and essentially always declining numbers of combatants.

4. As pointed out in Section 1.6, the state variables describe the system state, which is the minimum amount of information that allows one to predict the system's future from the past. This point is crucial for applications, since it forms the conceptual basis for <u>formulating</u> differential combat models. In other words, the state variables are the significant variables for describing and predicting, for example, the future evolution of the combat attrition process.

5. FELLER [25, p. 369] has said, "Conceputally, a MARKOV process is the probabilistic analogue of the processes of classical mechanics, where the future development is completely determined by the present state and is independent of the way in which the present state has developed. The processes of mechanics are in contrast to processes with after effect (or hereditary processes), such as occur in the theory of plasticity, where the whole past history of the system influences its future."
We observe that all the LANCHESTER-type combat processes considered in this book are equivalent to the processes of classical mechanics in the sense of not containing any hereditary effects.

6. For example, in a fixed-force-level-breakpoint battle (cf. Section 2.8 and also Chapter 3) there will be $(m_0 + 1 - m_{BP}) \times (n_0 + 1 - n_{BP})$ essential components in the state-probability vector, and corresponding to each such component is a differential-difference equation for its probabilistic evolution. Here m_{BP} denotes X's fixed force-level breakpoint, and similarly for n_{BP}.

7. There are also backward KOLMOGOROV equations (e.g. see FELLER [25, pp. 426-427]), but these are not of interest to us here.

8. The reader will find it instructive to show that such terms make no contribution to our final result (4.3.8), since $\lim_{\Delta t \to 0} O((\Delta t)^2)/\Delta t = 0$.

9. The (continuous-parameter MARKOV-chain) stochastic process corresponding to LANCHESTER's equations for modern warfare (2.2.1) has been called by B. O. KOOPMAN [55, p. 869] the LANCHESTER stochastic process. Here we have extended such terminology to include any stochastic-process version of LANCHESTER-type combat equations.

10. However, we may always use finite-difference techniques (see Appendix E) to numerically compute (with the aid of some type of automated computational device) approximate results for any specific battle.

11. We will omit any further explicit reference to the result (4.7.8), since it always holds. The reader should not forget this fact.

12. A few additional methods for computing the joint probability distribution of the numbers of survivors are given by KOOPMAN [55, pp. 863-866]. The author knows of no application of any of these additional methods to LANCHESTER combat theory. One significant omission by KOOPMAN [55] is the use of Monte Carlo methods, which are discussed in detail in Section 4.14 below. Although KOOPMAN [55, pp. 866-867] does talk about Monte Carlo simulations, he is using the term "Monte Carlo" in the sense of a modelling approach (see Chapter 1) and not in the sense of a computational procedure for estimating the state probabilities or some other statistical measure of battle outcome.

13. For a very readable introduction to difference equations, see HILDEBRAND [38].

14. The relationship between BROWN's [14] general result and CLARK's [16] hybrid method for the same general model has apparently never been explored.

15. See MORSE and KIMBALL [65, p. 70] for an example of such a calculation being carried out for the F|F attrition process with $m_0 = 5$ and $n_0 = 3$ for the special case in which $a = b$. Nevertheless (as we discuss below in the main text), no general analytical result that holds for all m_0 and $n_0 > 0$ is known to this author. It is interesting to note that although the CHAPMAN-KOLMOGOROV equation for the autonomous model (4.7.2) through (4.7.8) expresses the semi-group property of the state probabilities (see KOOPMAN [55, pp. 862-863]) and consequently the state-probability vector must be expressible in terms of a matrix-exponential function (e.g. see

BELLMAN [9, pp. 173-174])(i.e. each state probability is the sum of appropriately weighted exponential terms), the explicit analytical representation of the latter can be a formidable task (although trivial to represent symbolically) as attested to by the lack of analytical results for the state probabilities (cf. the analytical results for heterogeneous-force models in Section 7.8). Furthermore, CLARK's hybrid results (4.9.2) through (4.9.4) may be considered to be a manifestation of this latter fact of expressibility in terms of the matrix exponential.

16. Recently, GOLDIE [31] has given an analytical result that represents $P(t,m,n)$ as a double summation of negative exponentials similar to CLARK's [16] general result (4.9.2) but with undetermined coefficients (for which generating functions have been given, though). Moreover, GOLDIE [31, p. 606] has inferred that the result (4.9.35) was first given by GYE and LEWIS (see [34]) in 1974, whereas in reality ISBELL and MARLOW's [40] more general result (4.9.22) dates back to 1956. GOLDIE [31, p. 608] also attributes (4.10.21) and (4.10.22) to GYE and LEWIS [34] and makes no mention at all to the earlier work of D. G. SMITH [75].

17. A related heuristic discussion of the mechanisms of convective transport and diffusion of probability is to be found in Section 4.15 below.

18. From this discussion the reader should be able to geometrically visualize the relationship of the joint probability distribution to the win probabilities for such battles: the probability that one side wins is simply the total amount of probability that is "absorbed" onto the appropriate axis.

19. Except for the results of KISE and HIROSE [30] for the F|FT stochastic LANCHESTER-type attrition process, all the results that have appeared in the open literature have been for a fight to the finish.

20. Here, as for the F|F attrition process, all the results that have previously appeared in the open literature have been for a fight to the finish (see also Footnote 19 above).

21. These arguments, which are identical to those usually used to develop the negative binomial distribution for the number of independent BERNOULLI trials required to achieve a given number of successes (e.g. see FELLER [25, p. 155], go as follows. Denote the probability that the next casualty is an X casualty as $p = a/(a+b) = P^X_{NC}(m,n)$ and similarly $q = 1 - p = P^Y_{NC}(m,n)$. The probability that there are exactly $n_0 - n_{BP} - 1$ Y casualties out of a total of $m_0 + n_0 - m - n_{BP} - 1$ total casualties is given by

$$\binom{m_0 + n_0 - m - n_{BP} - 1}{n_0 - n_{BP} - 1} p^{m_0 - m} q^{n_0 - n_{BP} - 1}$$

The event that X wins with m final survivors occurs if (and only if) the battle state $(m, n_{BP} + 1)$ has been reached, i.e. exactly $n_0 - n_{BP} - 1$ Y casualties out of a total of $m_0 + n_0 - m - n_{BP} - 1$ casualties, and the next casualty is a Y one, with the corresponding probability of occurrence begin given by

$$\left\{\binom{m_0 + n_0 - m - n_{BP} - 1}{n_0 - n_{BP} - 1} p^{m_0 - m} q^{n_0 - n_{BP} - 1}\right\} q$$

whence follows (4.10.9). More formally, one could invoke (4.9.33) with

$$P^T_{m,n_{BP}+1}(m_0,n_0) = \binom{m_0 + n_0 - m - n_{BP} - 1}{n_0 - n_{BP} - 1} p^{m_0 - m} q^{n_0 - n_{BP} - 1}$$

and $P^Y_{NC}(m, n_{BP} + 1) = q$.

22. Professor G. E. LATTA has generously privately communicated the following proof of (4.10.24) to the author.

$$\sum_{m=m_{BP}+1}^{j} \frac{mj^{-m}}{(j-m)!} = \sum_{k=0}^{j-m_{BP}-1} \frac{(j-k)j^{-(j-k)}}{k!}$$

$$= \frac{1}{j^j} \sum_{k=0}^{j-m_{BP}-1} \left\{\frac{j^{k+1}}{k!} - \frac{j^k}{(k-1)!}\right\}$$

$$= \frac{1}{j^j} \left\{\frac{j^{(j-m_{BP})}}{(j-m_{BP}-1)!}\right\} = \frac{j^{-m_{BP}}}{(j-m_{BP}-1)!} \quad .$$

23. The reader will recognize that the situation here is exactly analogous to that concerning the practical usefulness of exact analytical expressions for the state-probability vector (i.e. the joint probability distribution for the number of survivors on each side).

24. For example, when $m_0 = n_0 = 30$, we have $(m_0 + n_0)! \approx 1.39 \times 10^{80}$ $<< m_0^{(m_0+n_0)}$, and such a computed numerical result will cause an overflow on, for example, the IBM 360/67 computer.

25. One could, of course, use a large-scale digital computer to compute for "enough" cases the probability of winning directly from the fundamental partial-difference equation (4.10.4) and to determine from this "data" the functional relationship between the battle-outcome probability distribution and the battle's parameters. Exactly how to do this and how to determine how many cases are "enough" are unanswered questions that doom this approach to failure. Thus, although it is easy to compute "by brute force" the probability of winning for specific numerical values of the battle parameters, the parametric determination of the probabilistic relation between battle inputs and outputs must apparently depend on having available simplifying approximations to the exact analytical results for the probability of winning.

26. In the second case of the F|F attrition process, however, the approximation applies to only a fight to the finish (i.e. a fixed-force-level-breakpoint battle in which $x_{BP} = y_{BP} = 0$).

27. Both BROWN [14; 15] and G. H. WEISS [89] give their results for a fight to the finish. Our result (4.11.5) for a fixed-force-level-breakpoint battle follows from (4.10.14) and consideration of their results (also, see below in main text).

28. All modern computing systems contain algorithms for generating samples of such unit uniform variates (e.g. see NAYLOR et al. [66, Chapter 3], EVANS, WALLACE, and SUTHERLAND [21, pp. 187-189], or FISHMAN [26, Chapter 7]).

29. Here we have defined the DIRAC delta function as a so-called ideal function or distribution (e.g. see COURANT and HILBERT [18, pp. 766-798] or FRIEDMAN [27] for further details).

30. Previously, researchers have always stated that the approximation is "good" as long as there is "little probability" that either side is annihilated, which is quite different from the correct statement that it is "good" as long as there is "little probability" that either side has reached its breakpoint.

31. This statement definitely appears to hold for continuous-time MARKOV-chain models in which the times between casualties are exponentially distributed. There has been some computational evidence, however, that in other cases (e.g. some other distribution for the times between casualties) this is not always true. Thus, without the appropriate qualifications being observed, it is simply not true that such a deterministic model invariably yields the same results for the mean course of combat as does a corresponding stochastic attrition model (e.g. a Monte Carlo simulation). More generally, as the author's colleague Professor C. J. ANCKER of the University of Southern California

has emphasized [3] to him, it is generally not true that a so-called mean-value model (obtained by replacing a random variable in a stochastic model by its mean value) yields a good approximation to the mean value of the corresponding stochastic process. Hopefully, we will see further clarification of this important point in the literature in the future.

32. As pointed out by TAYLOR and PARRY [79, p. 527] for the $(F+T)|(F+T)$ attrition process and a fixed-force-ratio-breakpoint battle, the condition of parity (i.e. neither side can win) between forces is an unstable equilibrium point. The author conjectures that this situation holds in general and leads to maximum dispersion of combat results under such initial conditions for stochastic LANCHESTER-type models.

33. These two attrition processes were treated in different ways by MORSE and KIMBALL [65, pp. 67-71]. For the $F|F$ LANCHESTER-type stochastic attrition process, the complete system of forward KOLMOGOROV equations were explicitly solved in one special case of numerical input values, and the average force levels at the end of battle (one side or the other annihilated) computed from these results [65, pp. 70-71]. For the $FT|FT$ LANCHESTER-type stochastic attrition process, only the random walk in the state space (corresponding to the state equation which in this case is the linear law) was considered [65, pp. 67-69].

34. SNOW [76, pp. 23-27] considered only the $(F+T)|(F+T)$ LANCHESTER-type stochastic attrition process (and its important special case, the $F|F$ attrition process) in his work. In other words, the $FT|FT$ LANCHESTER-type

stochastic attrition process was not investigated by him at all. Moreover, for the F|F stochastic attrition process, no new computational results were presented by SNOW [76] which are not to be already found in MORSE and KIMBALL [65, pp. 67-71].

35. From the numerical results presented by SPRINGALL [77, pp. 151-166], it is apparent that "large" here means more than 30 or 40 combatants on each side (in contrast to CLARK's [16, pp. 133-134, p. 137, and p. 243] in which "large" is taken to mean 100,000 or more combatants on each side).

36. Concering stochastic LANCHESTER-type combat models themselves, the following papers have appeared in the open literature subsequent to the work of SPRINGALL [77] and CLARK [16]: GRUBBS and SHUFORD [33], MARMA and DEUTSCH [60], MJELDE [64], JAIN and NAGABHUSHANAM [41], SHUFORD and GRUBBS [74], WATSON [81], GYE and LEWIS [34], and GOLDIE [31]. Mention should also be made of the reports by WEALE [82-86], JENNINGS [42-43], WEALE and PERYER [87], and KARR [45-50] and the M.Sc. thesis by GRAINGER [32]. In [50] KARR has reviewed and critiqued the seven DOAE reports [42-43; 82-85; 87].

37. Consideration of national security (i.e. classified material) is not a reason for this state of affairs, since there do exist classified channels of information dissemination. Along these lines, C. J. ANCKER [2] has observed that although analysis of military operations as a

basis for many types of expensive decisions consumes large amounts of time and money every year in U. S. military establishments, relatively little of the attention is focussed on actual combat (as opposed to combat-support operations), and even less on mathematical analysis of combat (as opposed to computer simulation of combat).

REFERENCES for Chapter 4

1. M. Abramowitz and I. A. Stegun (Editors), Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series, No. 55, Washington, D.C., 1964.

2. C. J. Ancker, "Review of the Book Force-on-Force Attrition Modelling," Phalanx 12, No. 4, 3 (1979).

3. C. J. Ancker, private communication, November 1979.

4. J. Andrighetti, "A Model for the Statistical Analysis of Land Combat Simulation and Field Experimentation Data," M.S. Thesis in Operations Research, Naval Postgraduate School, Monterey, California, September 1973 (AD 769 387).

5. L. Arnold, Stochastic Differential Equations: Theory and Applications, John Wiley, New York, 1974.

6. M. S. Bartlett, "Deterministic and Stochastic Models for Recurrent Epidemics," pp. 81-109 in Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman (Editor), University of California Press, Berkeley and Los Angeles, 1956.

7. M. S. Bartlett, "On Theoretical Models for Competitive and Predatory Biological Systems," Biometrika 44, 27-42 (1957).

8. M. S. Bartlett, Stochastic Population Models, Methuen, London, 1960.

9. R. E. Bellman, Introduction to Matrix Analysis, McGraw-Hill, New York, 1960.

10. A. T. Bharucha-Reid, Elements of the Theory of Markov Processes and Their Applications, McGraw-Hill, New York, 1960.

11. S. Bonder, "An Overview of Land Battle Modelling in the U.S.," pp. 73-88 in Proceedings of the Thirteenth Annual U.S. Army Operations Research Symposium, Fort Lee, Virginia, 1974.

12. K. C. Bowen, "Mathematical Battles," Bulletin of the Institute of Mathematics and Its Applications 9, 310-315 (1973).

13. F. C. Brooks, "The Stochastic Properties of Large Scale Battle Models, Opns. Res. 13, 1-17 (1965).

14. R. H. Brown, "A Stochastic Analysis of Lanchester's Theory of Combat," ORO-T-323, Operations Research Office, The Johns Hopkins University, Chevy Chase, Maryland, December 1955 (AD 82 944).

15. R. H. Brown, "Theory of Combat: The Probability of Winning," Opns. Res. 11, 418-425 (1963).

16. G. M. Clark, "The Combat Analysis Model," Ph.D. Thesis, The Ohio State University, Columbus, Ohio, 1969.

17. G. M. Clark, "The Combat Analysis Model," Chapter 11 in "The Tank Weapon System," A. B. Bishop and G. M. Clark (Editors), Report No. AR 69-2B, Systems Research Group, The Ohio State University, Columbus, Ohio, September 1969.

18. R. Courant and D. Hilbert, Methods of Mathematical Physics, Volume II, Interscience, New York, 1962.

19. J. D. Craig, "The Effect of Uncertainty on Lanchester-Type Equations of Combat," M.S. Thesis in Operations Research, Naval Postgraduate School, Monterey, California, September 1975 (AD A017 550).

20. D. O. Etter, "Deterministic Combat Attrition Models for Spatially Distributed Forces," P-577, Institute for Defense Analyses, Arlington, Virginia, May 1971.

21. G. W. Evans, G. F. Wallace, and G. L. Sutherland, Simulation Using Digital Computers, Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

22. R. L. Farrell and R. J. Freedman, "Investigations of the Variation of Combat Model Predictions with Terrain Line of Sight," Report No. AMSAA-1, FR75-1, Vector Research, Inc., Ann Arbor, Michigan, January 1975.

23. W. Feller, "On the Normal Approximation to the Binomial Distribution," Ann. Math. Stat. 16, 319-329 (1945).

24. W. Feller, "Diffusion Processes in Genetics," pp. 227-246 in Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman (Editor), University of California Press, Berkeley and Los Angeles, 1951.

25. W. Feller, An Introduction to Probability Theory and Its Applications, Volume I (Second Edition), John Wiley, New York, 1957.

26. G. S. Fishman, Concepts and Methods in Discrete Event Digital Simulation, John Wiley, New York, 1973.

27. A. Friedman, Generalized Functions and Partial Differential Equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

28. P. R. Garabedian, Partial Differential Equations, John Wiley, New York, 1964.

29. I. I. Gihman and A. V. Skorohod, Stochastic Differential Equations, Springer-Verlag, New York, Heidelberg, Berlin, 1972.

30. N. S. Goel, S. C. Maitra, and E. W. Montroll, "On the Volterra and Other Nonlinear Models of Interacting Populations," Rev. Mod. Phys. 43, 231-276 (1971) (also published as a monograph with the same title by Academic Press, New York, 1971).

31. C. M. Goldie, "Lanchester Square-Law Battles: Transient and Terminal Distributions," J. Appl. Prob. 14, 604-610 (1977).

32. P. L. Grainger, "The Use of a Stochastic Tank Engagement Model to Examine the Effects of Moving Target Correlation and to Derive Equivalent Lanchester Equations," M. Sc. Thesis in Operational Research, Department of Statistics and OR, Brunel University, Uxbridge, United Kingdom, June 1976.

33. F. E. Grubbs and J. H. Shuford, "A New Formulation of Lanchester Combat Theory," Opns. Res. 21, 926-941 (1973).

34. R. Gye and T. Lewis, "Lanchester's Equations" Mathematics and the Art of War, A Historical Survey and Some New Results," Math. Scientist 1, 107-119 (1976).

35. P. J. Haysman and B. E. Mortagy, "References on the Lanchester Theory of Combat to 1980," Working Paper OR/WP/6, Department of Management Sciences, The Royal Military College of Science, Shrivenham, United Kingdom, January 1980.

36. F. B. Hildebrand, Advanced Calculus for Engineers, Prentice-Hall, Englewood Cliffs, 1948.

37. F. B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill, New York, 1956.

38. F. B. Hildebrand, Finite-Difference Equations and Simulations, Prentice-Hall, Englewood Cliffs, 1968.

39. D. L. Iglehart, "Diffusion Approximations in Applied Probability," pp. 235-254 in Mathematics of the Decision Sciences, Part 2, G. B. Dantzig and A. F. Veinott (Editors), American Mathematical Society, Providence, Rhode Island, 1968.

40. J. R. Isbell and W. H. Marlow, "Attrition Games," Naval Res. Log. Quart. 3, 71-94 (1956).

41. G. C. Jain and A. Nagabhushanam, "A Two-State Markovian Correlated Combat," Opns. Res. 22, 440-444 (1974).

42. N. Jennings, "The Mathematics of Battle III. Approximate Moments of the Distribution of States of a Simple Heterogeneous Battle," M7315, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, June 1973.

43. N. Jennings, "The Mathematics of Battle IV. Stochastic 'Linear Law' Battles," M7316, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, June 1973.

44. S. Karlin, A First Course in Stochastic Processes, Academic Press, New York, 1966.

45. A. F. Karr, "Stochastic Attrition Models of Lanchester Type," P-1030, Institute for Defense Analyses, Arlington, Virginia, June 1974.

46. A. F. Karr, "A Generalized Stochastic Lanchester Attrition Process," P-1080, Institute for Defense Analyses, Arlington, Virginia, September 1975.

47. A. F. Karr, "On Simulations of the Stochastic, Homogeneous, Lanchester Square-Law Attrition Processes," P-1112, Institute for Defense Analyses, Arlington, Virginia, September 1975.

48. A. F. Karr, "On Simulations of the Stochastic, Homogeneous, Lanchester Linear-Law Attrition Process," P-1113, Institute for Defense Analyses, Arlington, Virginia, September 1975.

49. A. F. Karr, "A Class of Lanchester Attrition Processes," P-1230, Institute for Defense Analyses, Arlington, Virginia, December 1976.

50. A. F. Karr, "A Review of Seven DOAE Papers on Mathematical Models of Attrition," P-1249, Institute for Defense Analyses, Arlington, Virginia, November 1977.

51. D. G. Kendall, "Stochastic Processes and Population Growth," J. Roy. Stat. Soc., Series B 11, 230-264 (1949).

52. M. G. Kendall, The Advanced Theory of Statistics, Volume 1, Charles Griffin & Company Limited, London, 1945.

53. T. Kisi and T. Hirose, "Winning Probability in an Ambush Engagement," Opns. Res. 14, 1137-1138 (1966).

54. L. Kleinrock, Queueing Systems, Volume I: Theory, John Wiley, New York, 1975.

55. B. O. Koopman, "A Study of the Logical Basis of Combat Simulation," Opns. Res. 18, 855-882 (1970).

56. F. W. Lanchester, "Aircraft in Warfare: The Dawn of the Fourth Arm - No. V., The Principle of Concentration," Engineering 98, 422-423 (1914), (reprinted on pp. 2138-2148 of the World of Mathematics, Vol. IV, J. Newman (Editor), Simon and Schuster, New York, 1956).

57. W. U. Lee and A. Wannasilpa, "Comparison of a Deterministic and a Stochastic Model for the Probability of Winning in a Two-Sided Combat Situation," M.S. Thesis in Operations Research, Naval Postgraduate School, Monterey, California, September 1972 (AD 756 536).

58. D. Ludwig, Stochastic Population Theories, Vol. 3 in Lecture Notes in Biomathematics, Springer-Verlag, Berlin, Heidelberg, and New York, 1974.

59. A. Maradudin and G. H. Weiss, "A Study of Some Lanchester-Like Equations," Tech. Report No. 95, Physics Dept., University of Maryland, College Park, Maryland, 1958 (AD 154 168).

60. V. Marma and K. Deutsch, "Survival in Unfair Conflict: Odds, Resources, and Random Walk Models," Behavioral Science 18, 313-334 (1973).

61. C. Marshall, "Probabilistic Models in the Theory of Combat," Trans. New York Acad. Sciences, Series II, 27, 477-487 (1965).

62. D. D. McCracken and W. S. Dorn, Numerical Methods and FORTRAN Programming, John Wiley, New York, 1964.

63. W. E. Milne, Numerical Solution of Differential Equations, John Wiley, New York, 1953.

64. K. M. Mjelde, "An Analytical Approach to a Class of Battles," Opns. Res. 22, 93-99 (1974).

65. P. M. Morse and G. E. Kimball, Methods of Operations Research, The M.I.T. Press, Cambridge, Massachusetts, 1951.

66. T. H. Naylor, J. L. Balintfy, D. S. Burdick, and K. Chu, Computer Simulation Techniques, John Wiley, New York, 1966.

67. L. Padulo and M. A. Arbib, System Theory, Saunders, Philadelphia, 1974.

68. E. Parzen, Modern Probability Theory and Its Applications, John Wiley, New York, 1960.

69. E. Parzen, Stochastic Processes, Holden-Day, San Francisco, 1962.

70. K. Pearson, "On the Applications of the Double Bessel Function $\mathcal{K}_{\tau_1,\tau_2}(x)$ to Statistical Problems," Biometrika 25, 158-178 (1933).

71. K. Pearson (Editor), Tables of the Incomplete Beta-Function, Cambridge University Press, Cambridge, 1934.

72. P. P. Perla and J. P. Lehoczky, "A New Approach to the Analysis of Stochastic Lanchester Processes: I. Time Evolution," Tech. Report No. 135, Department of Statistics, Carnegie-Mellon University, Pittsburgh, Pennsylvania, September 1977 (AD A045 176).

73. E. C. Pielou, An Introduction to Mathematical Ecology, Wiley-Interscience, New York, 1969.

74. J. H. Shuford and F. E. Grubbs, "Stopping Rules for War Games or Combat Simulations with Exponential Life Times," Opns. Res. 23, 824-829 (1975).

75. D. G. Smith, "The Probability Distribution of the Number of Survivors in a Two-Sided Combat Situation," Operational Res. Quart. 16, 429-437 (1965).

76. R. N. Snow, "Contributions to Lanchester Attrition Theory," Report RA-15078, The RAND Corporation, Santa Monica, California, April 1948.

77. A. Springall, "Contributions to Lanchester Combat Theory," Ph.D. Thesis, Virginia Polytechnic Institute, Blacksburg, Virginia, March 1968.

78. J. G. Taylor, "Application of Differential Games to Problems of Military Conflict: Tactical Allocation Problems - Part II," Tech. Report NPS 55Tw72111A, Naval Postgraduate School, Monterey, California, November 1972 (AD 758 663).

79. J. G. Taylor and S. H. Parry, "Force-Ratio Considerations for Some Lanchester-Type Models of Warfare," Opns. Res. 23, 522-533 (1975).

80. J. Todd (Editor), Survey of Numerical Analysis, McGraw-Hill, New York, 1962.

81. R. K. Watson, "An Application of Martingale Methods to Conflict Models," Opns. Res. 24, 380-382 (1976).

82. T. G. Weale, "The Mathematics of Battle I. A Bivariate Probability Distribututicn," M7129, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, December 1971.

83. T. G. Weale, "The Mathematics of Battle II. The Moments of the Distribututlon of Battle States," M7130, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, October 1972.

84. T. G. Weale, "The Mathematics of Battle V. Homogeneous Battles with General Attrition Functions," M7511, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, August 1975.

85. T. G. Weale, "The Mathematics of Battle VI. The Distribution of the Duration of Battle," M76126, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, June 1976.

86. T. G. Weale, "Mathematics of Battle VIII. The Heterogeneous Battle Model," M78106, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, April 1978.

87. T. G. Weale and E. Peryer, "The Mathematics of Battle VII. Moments of the Distribututlon of States for a Battle with General Attrition Functions," M77105, Defence Operational Analysis Establishment, West Byfleet, United Kingdom, January 1977.

88. S. Weintraub, Tables of the Cumulative Binomial Probability Distribution for Small Values of p, The Free Press of Glencoe (A Division of The Macmillan Company), New York, 1963.

89. G. H. Weiss, "Comparison of a Deterministic and a Stochastic Model for Interaction Between Antagonistic Species," Biometrics 19, 595-602 (1963).

90. G. H. Weiss, "On the Spread of Epidemics by Carriers," Biometrics 21, 481-490 (1965).

91. D. Willard, "Lanchester as Force in History: An Analysis of Land Battles of the Years 1618-1905," RAC-TP-74, Research Analysis Corporation, Bethesda, Maryland, November 1962 (AD 297 375).

*APPENDIX C: SOLUTION OF THE FUNDAMENTAL PARTIAL-DIFFERENCE EQUATION FOR THE PROBABILITY THAT X WINS WITH m FINAL SURVIVORS $P_{m,n_{BP}}(m_0,n_0)$

1. Introduction.

In this appendix we will show how to develop the analytical solution to the fundamental partial-difference equation for the probability that X wins with m final survivors $P_{m,n_{BP}}(m_0,n_0)$ for the stochastic FT|FT and F|F LANCHESTER-type attrition processes. We will also consider the analogous development for the probability that the course of the battle passes through the transient state (m,n) at some time during the battle for the F|F attrition process. In all cases we will consider only a fixed-force-level-breakpoint battle. Not only are these results of interest in their own right, but the analytical-solution approaches presented here should be of use for solving other partial-difference equations that arise in the LANCHESTER theory of combat, e.g. the partial-difference equation for the coefficients in CLARK's [5] analytical representation of the state-probability vector (4.9.2) [see equations (4.9.5) through (4.9.8) above]. We will show how to solve such partial-difference equations by using both generating functions and also a separation-of-variables approach.

2. Development for the FT|FT Attrition Process.

For the FT|FT stochastic LANCHESTER-type attrition process, the fundamental partial-difference equation for the probability that X wins a fixed-force-level-breakpoint battle with m final survivors $P_{m,n_{BP}}(m_0,n_0)$ reads for $m_0 \geq m > m_{BP}$ and $n_0 > n_{BP}$

$$P_{m,n_{BP}}(m_0,n_0) = pP_{m,n_{BP}}(m_0-1,n_0) + qP_{m,n_{BP}}(m_0,n_0-1) \qquad \text{(C.1)}$$

with boundary conditions

$$P_{m,n_{BP}}(m_0,n_{BP}) = \begin{cases} 1 & \text{for } m_0 = m, \\ 0 & \text{for } m_0 > m, \end{cases}$$

and (C.2)

$$P_{m,n_{BP}}(m-1,n_0) = 0 \qquad \text{for } n_0 \geq n_{BP} .$$

Here, for convenience we have let

$$p = \frac{a}{a+b} \quad \text{and} \quad q = 1 - p = \frac{b}{a+b} . \qquad \text{(C.3)}$$

The above partial-difference equation (C.1) with boundary condition (C.2) is most conveniently analytically solved (by nonprobabilistic methods) by using the generating-function approach (e.g. see HILDEBRAND [8] for further details) as follows. First, we observe that $P_{m,n_{BP}}(m_0,n_0) = 0$ when either $m_0 < m$ or $n_0 < n_{BP}$, since it is impossible for either side to have a final force level greater than its initial one. Introducing the generating function

$$G(n_0,s) = \sum_{m_0=m}^{\infty} s^{m_0} P_{m,n_{BP}}(m_0,n_0) , \qquad \text{(C.4)}$$

we can multiply (C.1) by s^{m_0} and sum over m_0 from m to ∞ to find after some straightforward manipulations that the fundamental partial-difference

equation with its associated boundary conditions yields the following ordinary-difference equation for the generating function: for $n_0 > n_{BP}$

$$G(n_0,s) = \left(\frac{q}{1 - ps}\right) G(n_0-1,\ s)\ , \tag{C.5}$$

with initial condition

$$G(n_{BP},\ s) = s^m\ . \tag{C.6}$$

Solving the above difference equation (C.5) with initial condition (C.6), we find that the generating function $G(n_0,s)$ is given by

$$G(n_0,s) = \left(\frac{q}{1 - ps}\right)^{n_0-n_{BP}} s^m\ . \tag{C.7}$$

To obtain $P_{m,n_{BP}}(m_0,n_0)$ from its generating function $G(n_0,s)$ given by (C.7), we recall (C.4) and observe that

$$P_{m,n_{BP}}(m_0,n_0) = \frac{1}{(m_0)!} \frac{d^{m_0}}{ds^{m_0}} G(n_0,s)\bigg|_{s=0}\ . \tag{C.8}$$

Recalling LEIBNITZ's rule that [7]

$$\frac{d^N}{ds^N}(uv) = \sum_{k=0}^{N} \binom{N}{k} \frac{d^{N-k}u}{ds^{N-k}} \frac{d^k v}{ds^k}\ ,$$

we see that

$$\frac{d^{m_0}}{ds^{m_0}} G(n_0,s) = q^{n_0-n_{BP}} \sum_{k=0}^{m_0} \binom{m_0}{k} \left\{ \frac{d^{m_0-k}}{ds^{m_0-k}} (1-ps)^{-(n_0-n_{BP})} \right\} \frac{d^k s^m}{ds^k} \quad . \tag{C.9}$$

Let us also observe that for $N \geq 0$

$$\frac{d^N}{ds^N} (1 - ps)^{-(n_0-n_{BP})} \Bigg|_{s=0} = p^N \frac{(n_0 - n_{BP} + N - 1)!}{(n_0 - n_{BP} - 1)!} \quad . \tag{C.10}$$

It follows from (C.8), (C.9), and (C.10) that for $m_0 \geq m > m_{BP}$ and $n_0 \geq n_{BP}$

$$P_{m,n_{BP}}(m_0,n_0) = \frac{q^{n_0-n_{BP}}}{(m_0)!} \binom{m_0}{m_0-m} p^{m_0-m} \frac{(m_0 + n_0 - m - n_{BP} - 1)!}{(n_0 - n_{BP} - 1)!} m! \ ,$$

or

$$P_{m,n_{BP}}(m_0,n_0) = \binom{m_0 + n_0 - m - n_{BP} - 1}{n_0 - n_{BP} - 1} p^{m_0-m} q^{n_0-n_{BP}} \quad . \tag{C.11}$$

Using (C.3), we finally obtain

$$P_{m,n_{BP}} = \binom{m_0 + n_0 - m - n_{BP} - 1}{n_0 - n_{BP} - 1} \left(\frac{a}{a + b}\right)^{m_0-m} \left(\frac{b}{a + b}\right)^{n_0-n_{BP}} \ , \tag{C.12}$$

which appears in the main text as (4.10.9).

3. Development for the F|F Attrition Process.

For the F|F attrition process, the fundamental partial-difference equation for $P_{m,n_{BP}}(m_0,n_0)$ reads for $m_0 \geq m > m_{BP}$ and $n_0 > n_{BP}$

$$P_{m,n_{BP}}(m_0,n_0) = \left(\frac{n_0}{\frac{b}{a} m_0 + n_0}\right) P_{m,n_{BP}}(m_0-1,\ n_0)$$

$$+ \left(\frac{\frac{b}{a} m_0}{\frac{b}{a} m_0 + n_0}\right) P_{m,n_{BP}}(m_0,n_0-1) \ , \tag{C.13}$$

with boundary conditions (C.2). Unfortunately, the generating-function approach used above no longer very conveniently yields the solution to this partial-difference equation because of the complexity of its coefficients.

Following R. H. BROWN [3; 4], we will use the approach of separation of variables to solve the fundamental partial-difference equation (C.13). Accordingly, we assume that $P_{m,n_{BP}}(m_0,n_0)$ has the form

$$P_{m,n_{BP}}(m_0,n_0) = g(m_0)\ h(n_0)\ F(m,n_{BP}) \ . \tag{C.14}$$

It follows that for $m_0 \geq m > m_{BP}$ and $n_0 > n_{BP}$

$$- \left\{\frac{g(m_0-1) - g(m_0)}{m_0 g(m_0)}\right\} = \frac{b}{a} \left\{\frac{h(n_0-1) - h(n_0)}{n_0 h(n_0)}\right\} \ . \tag{C.15}$$

Since the left-hand side of (C.15) is independent of n_0 and the right-hand side is independent of m_0, they each must be equal to the same constant value (independent of both m_0 and n_0). Let us call this constant value

$1/\lambda$ and write for $m_0 \geq m > m_{BP}$ and $n_0 > n_{BP}$

$$- \left\{ \frac{g(m_0-1) - g(m_0)}{m_0 g(m_0)} \right\} = \frac{b}{a} \left\{ \frac{h(n_0-1) - h(n_0)}{n_0 h(n_0)} \right\} = \frac{1}{\lambda} \; . \tag{C.16}$$

It follows that

$$g(m_0) = \left(\frac{-\lambda}{m_0 - \lambda} \right) g(m_0-1) \qquad \text{for} \quad m_0 \geq m \; , \tag{C.17}$$

and

$$h(n_0) = \left(\frac{\frac{b}{a}\lambda}{\frac{b}{a}\lambda + n_0} \right) h(n_0-1) \qquad \text{for} \quad n_0 > n_{BP} \; . \tag{C.18}$$

Hence, save for multiplicative factors that are arbitrary functions of period 1 (or "periodic constants") [2; 8; 11], we find that

$$g(m_0) = \frac{(-\lambda)^{m_0}}{\Gamma(m_0 - \lambda + 1)} \; , \tag{C.19}$$

and

$$h(n_0) = \frac{(\frac{b}{a}\lambda)^{n_0}}{\Gamma(\frac{b}{a}\lambda + n_0 + 1)} \; . \tag{C.20}$$

Let us now formally acknowledge the dependence of the g and h functions on λ and write

$$g = g(\lambda, m_0), \qquad h = h(\lambda, n_0), \qquad F = F(\lambda, m, n_{BP}) \; . \tag{C.21}$$

If $g(\lambda,n_0)\, h(\lambda,n_0)\, F(\lambda,m,n_{BP})$ satisfies the linear partial-difference equations (C.13), then so will $\sum_{\lambda \in S_\lambda} g(\lambda,m_0)\, h(\lambda,n_0)\, F(\lambda,m,n_{BP})$ for any

finite set of values for λ (here denoted as S_λ). Hence, we will assume a solution of the form

$$P_{m,n_{BP}}(m_0,n_0) = \sum_{\lambda \in S_\lambda} g(\lambda,m_0)\ h(\lambda,n_0)\ F(\lambda,m,n_{BP}) \ , \tag{C.22}$$

and then we will try to choose S_λ and $F(\lambda,m,n_{BP})$ in order to satisfy the boundary condition (C.2). By construction then, we will have obtained the solution to our problem (well-known to be unique) once the boundary conditions (C.2) have been satisfied.

Thus, we will look for a solution of the form

$$P_{m,n_{BP}}(m_0,n_0) = \sum_{\lambda \in S_\lambda} \frac{(-\lambda)^{m_0} \left(\frac{b}{a}\lambda\right)^{n_0} F(\lambda,\ m,\ n_{BP})}{\Gamma(m_0 - \lambda + 1)\ \Gamma\left(\frac{b}{a}\lambda + n_0 + 1\right)} \ . \tag{C.23}$$

The second boundary condition of (C.2) that $P_{m,n_{BP}}(m-1,n_0) = 0$ for all $n_0 \geq n_{BP}$ yields that

$$\frac{1}{\Gamma(m - \lambda)} = 0 \ ,$$

whence $\lambda = m, m+1, m+2, \ldots$, since the gamma function is an analytic function except for isolated poles at 0 and integer points on the negative real axis in the complex plane [6, pp. 206-207]. Hence, λ takes on only integer values, and we will henceforth always replace λ by j. Observing that for $j = m_0+1,\ m_0+2,\ \ldots$

$$\frac{1}{\Gamma(m_0 - j + 1)} = 0 \ ,$$

we find that the second boundary condition of (C.2) has yielded that

$$P_{m,n_{BP}}(m_0,n_0) = \left(\frac{b}{a}\right)^{n_0} \sum_{j=m}^{m_0} \frac{(-1)^{m_0} j^{m_0+n_0} F(j, m, n_{BP})}{(m_0-j)!\ \Gamma(\frac{b}{a} j + n_0 + 1)} \quad . \tag{C.24}$$

Before invoking the first boundary condition of (C.2), however, it will be convenient to transform the expression (C.24) with $n_0 = n_{BP}$ to a more useful form. Thus letting $k = m_0 - j$, we find that for $n_0 = n_{BP}$ (C.24) becomes

$$P_{m,n_{BP}}(m_0,n_{BP}) = \left(\frac{b}{a}\right)^{n_{BP}} \frac{(-1)^{m_0}}{M!} \sum_{k=0}^{M} \binom{M}{k} (k + m)^{m_0+n_{BP}} \frac{k!\ F(j,m,n_{BP})}{\Gamma(\frac{b}{a} j + n_{BP} + 1)} \tag{C.25}$$

where $M = m_0 - m$. Further manipulations then yield that we may write (C.25) for $m_0 \geq m > m_{BP}$

$$P_{m,n_{BP}}(m_0,n_{BP}) = \frac{(-1)^M}{M!} \sum_{k=0}^{M} \binom{M}{k} (-1)^k (k + m)^M\ G(k,m,n_{BP}) \ , \tag{C.26}$$

where

$$G(k,m,n_{BP}) = \left(\frac{b}{a}\right)^{n_{BP}} \frac{(-1)^j j^{m+n_{BP}} (j-m)!\ F(j,m,n_{BP})}{\Gamma(\frac{b}{a} j + n_{BP} + 1)} \quad . \tag{C.27}$$

Using the above result (C.26) for $P_{m,n_{BP}}(m_0, n_{BP})$, we find that the first boundary condition of (C.2) then yields

$$\sum_{k=0}^{M} \binom{M}{k} (-1)^k (k+m)^M\ G(k,m,n_{BP}) = \begin{cases} (-1)^M\ M! & \text{for } M = 0 \\ 0 & \text{for } M > 0. \end{cases} \tag{C.28}$$

According to Lemma C.1, which is stated and proven in the last section of this appendix, it follows that for $M > 0$ and $L \geq 1$

$$\sum_{k=0}^{M} \binom{M}{k} (-1)^k (k+m)^{M-L} = 0 . \tag{C.29}$$

In order that (C.29) holds for all $M > 0$, we must have $L = 1$, and hence for $M > 0$

$$G(k,m,n_{BP}) = \frac{C}{k+m} \tag{C.30}$$

will satisfy the lower condition on the right-hand side of (C.28). For $M = 0$, (C.28) yields that $C = m$ and hence

$$G(k,m,n_{BP}) = \frac{m}{k+m} \tag{C.31}$$

leads to the satisfying of the first boundary condition of (C.2). From (C.27) it then follows that

$$F(j,m,n_{BP}) = m\left(\frac{b}{a}\right)^{-n_{BP}} \frac{(-1)^j \, j^{-m-n_{BP}-1} \, \Gamma(\frac{b}{a} j + n_{BP} + 1)}{(j-m)!} , \tag{C.32}$$

whence

$$P_{m,n_{NP}} = m\left(\frac{b}{a}\right)^{n_0-n_{BP}} \sum_{j=m}^{m_0} \frac{(-1)^{m_0-j} \, j^{m_0+n_0-m-n_{BP}-1} \, \Gamma(\frac{b}{a} j + n_{BP} + 1)}{(m_0-j)! \, (j-m)! \, \Gamma(\frac{b}{a} j + n_0 + 1)}, \tag{C.33}$$

which appears in the main text as (4.10.21). We will refer to the above approach for solving the fundamental partial-difference equation for

$P_{m,n_{BP}}(m_0,n_0)$ as BROWN's separation-of-variables method. It finally should be noted that (C.33) may also be written for $m_0 \geq m > m_{BP}$ and $n_0 \geq n_{BP}$ as

$$P_{m,n_{BP}}(m_0,n_0) = \frac{m(-1)^M \left(\frac{b}{a}\right)^{N_{BP}}}{M!\ N_{BP}!} \sum_{k=0}^{M} \frac{(-1)^k \binom{M}{k} (k+m)^{M+N_{BP}-1}}{\binom{\frac{b}{a}(k+m) + n_{BP} + N_{BP}}{N_{BP}}} , \qquad \text{(C.34)}$$

where $k = j-m$, $M = m_0-m$, and $N_{BP} = n_0 - n_{BP}$.

4. Development of the Transient-State-Passage Probability $P_{m,n}(m_0,n_0)$ for the F|F Attrition Process.

For the F|F attrition process, the fundamental partial-difference equation for the probability that the battle passes through the transient state (m,n) at some time during the battle $P_{m,n}(m_0,n_0)$ reads for $m_0 \geq m > m_{BP}$ and $n_0 > n > n_{BP}$

$$P_{m,n}(m_0,n_0) = \left(\frac{n_0}{\frac{b}{a}m_0 + n_0}\right) P_{m,n}(m_0-1,n_0)$$

$$+ \left(\frac{\frac{b}{a}m_0}{\frac{b}{a}m_0 + n_0}\right) P_{m,n}(m_0,n_0-1) , \qquad \text{(C.35)}$$

with boundary conditions

$$P_{m,n}(m_0,n) = \begin{cases} 1 & \text{for } m_0 = m, \\ 1\Big/\left[\prod_{k=1}^{m_0-m} \{1 + (b/an)(k+m)\}\right] & \text{for } m_0 > m , \end{cases} \tag{C.36}$$

and

$$P_{m,n}(m-1,n_0) = 0 \qquad \text{for } n_0 \geq n_{BP} .$$

It should be observed that $P_{m,n}(m_0,n_0)$ satisfies the same fundamental partial-difference equation as does $P_{m,n_{BP}}(m_0,n_0)$ but that the boundary conditions differ for these two probabilities. In this respect, the reader should compare (C.2) with (C.36).

Since the fundamental partial-difference equation for $P_{m,n}(m_0,n_0)$ and its second boundary condition are the same as that for $P_{m,n_{BP}}(m_0,n_0)$, BROWN's separation-of-variables approach and use of the second boundary condition of (C.36) yield for $m_0 \geq m > m_{BP}$ and $n_0 \geq n > n_{BP}$

$$P_{m,n}(m_0,n_0) = \left(\frac{b}{a}\right)^{n_0} \sum_{j=m}^{m_0} \frac{(-1)^{m_0} j^{m_0+n_0} F(j,m,n)}{(m_0-j)!\ \Gamma(\frac{b}{a} j + n_0 + 1)} . \tag{C.37}$$

Again, it is convenient to rewrite (C.37) for $n_0 = n$ as

$$P_{m,n}(m_0,n) = \frac{(-1)^M}{M!} \sum_{k=0}^{M} \binom{M}{k} (-1)^k (k+m)^M G(k,m,n) , \tag{C.38}$$

where

$$M = m_0 - m \ ,$$

$$G(k,m,n) = \left(\frac{b}{a}\right)^n \frac{(-1)^j \ j^{m+n} \ (j-m)! \ F(j,m,n)}{\Gamma(\frac{b}{a} j + n + 1)} \ , \qquad \text{(C.39)}$$

and

$$k = j - m \ .$$

Let us also observe that the first boundary condition of (C.36) may be written as

$$P_{m,n}(m_0,n) = \begin{cases} 1 & \text{for } M = 0 \ , \\ 1 \Big/ \left[\prod_{k=1}^{M} \{1 + (b/an)(k + m)\} \right] & \text{for } M > 0 \ . \end{cases} \qquad \text{(C.40)}$$

It is then convenient to write (C.40) for $M \geq 0$ as

$$P_{m,n}(m_0,n) = \frac{1 + (b/an)m}{\prod_{k=0}^{M} \{1 + (b/an)(k + m)\}} \ . \qquad \text{(C.41)}$$

Applying the first boundary condition of (C.36) in the form (C.41) to (C.38), we find that

$$\sum_{k=0}^{M} \binom{M}{k} (-1)^k \ (k + m)^M \ H(k,m,n) = \frac{(-1)^M \ M!}{\prod_{k=0}^{M} \{1 + (b/an)(k + m)\}} \ , \qquad \text{(C.42)}$$

where

$$H(k,m,n) = n\left(\frac{b}{a}\right)^n \frac{(-1)^j\, j^{m+n}\,(j-m)!\, F(j,m,n)}{(\frac{b}{a} m + n)\, \Gamma(\frac{b}{a} j + n + 1)} . \tag{C.43}$$

It remains to find $H(k,m,n)$ such that (C.42) is satisfied. To do this, let us first observe that for $M = 0$, (C.42) yields that

$$H(0,m,n) = \frac{1}{1 + \{b/(an)\}m} , \tag{C.44}$$

which can then itself be used in (C.43) with $M = 1$ to show that

$$H(1,m,n) = \frac{1}{1 + \{b/(an)\}(m+1)} . \tag{C.45}$$

Using (C.44) and (C.45), we can then show with a somewhat lengthier calculation for $M = 2$ in (C.42) that

$$H(2,m,n) = \frac{1}{1 + \{b/(an)\}(m+2)} . \tag{C.46}$$

Thus, we are led to conjecture that

$$H(k,m,n) = \frac{1}{1 + \{b/(an)\}(m+k)} \tag{C.47}$$

will satisfy (C.42), and application of Lemma C.2 (which is stated and proven in the next section of this appendix) confirms this conjecture. From (C.44) it then follows that

$$F(j,m,n) = \left(\frac{b}{a} m + n\right)\left(\frac{b}{a}\right)^{-n} \frac{(-1)^j\, j^{-m-n}\, \Gamma(\frac{b}{a} j + n)}{(j-m)!} , \tag{C.48}$$

whence

$P_{m,n}(m_0,n_0)$

$$= \left(\frac{b}{a} m + n\right) \left(\frac{b}{a}\right)^{n_0-n} \sum_{j=m}^{m_0} \frac{(-1)^{m_0-j} \, j^{m_0+n_0-m-n} \, \Gamma(\frac{b}{a} j + n)}{(m_0-j)! \, (j-m)! \, \Gamma(\frac{b}{a} j + n_0 + 1)} , \tag{C.49}$$

which appears in the main text as (4.9.30). It finally should be noted that (C.49) may also be written as

$$P_{m,n}(m_0,n_0) = \frac{(\frac{b}{a} m + n)(-1)^M (\frac{b}{a})^N}{M! \, (N+1)!} \sum_{k=0}^{M} \frac{(-1)^k \binom{M}{k} (k+m)^{M+N}}{\binom{\frac{b}{a}\{k+m\} + n + N}{N+1}} , \tag{C.50}$$

where $k = j-m$, $M = m_0-m$, and $N = n_0-n$.

5. <u>Two Important Identities Used in Solving the Fundamental Partial-Difference Equations for the F|F Attrition Process</u>.

In this section we will state and prove two lemmas that we have invoked above in solving the fundamental partial-difference equations for $P_{m,n_{BP}}(m_0,n_0)$ and $P_{m,n}(m_0,n_0)$ for the F|F attrition process.

LEMMA C.1: For any integers M and $N \geq 0$ and real number α, we have that

$$\sum_{k=0}^{M} \binom{M}{k} (-1)^k (k+\alpha)^N = \begin{cases} 0 & \text{for } 0 \leq N < M , \\ (-1)^M M! & \text{for } N = M , \\ (-1)^M M! \, (\frac{M}{2} + \alpha) & \text{for } N = M+1 . \end{cases} \tag{C.51}$$

PROOF. Consider

$$e^{\alpha x}(e^x-1)^M = (-1)^M e^{\alpha x}(1-e^x)^M = (-1)^M e^{\alpha x} \sum_{k=0}^{M} \binom{M}{k} (-1)^k e^{kx} .$$

Consequently,

$$e^{\alpha x}(e^x-1)^M = (-1)^M \sum_{k=0}^{M} \binom{M}{k} (-1)^k e^{(k+\alpha)x} ,$$

whence follows

$$e^{\alpha x}(e^x-1)^M = (-1)^M \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=0}^{M} \binom{M}{k} (-1)^k (k+\alpha)^j . \tag{C.52}$$

We also have that

$$e^{\alpha x}(e^x-1)^M = e^{\alpha x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^M . \tag{C.53}$$

Observing that

$$\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^M = x^M\left(1 + \frac{x}{2} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots\right)^M,$$

we may write (C.53) as

$$e^{\alpha x}(e^x-1)^M = x^M\left(1 + \alpha x + \frac{\alpha^2 x^2}{2!} + \cdots\right)\left(1 + \frac{M}{2} x + \frac{M(3M+1)}{4} x^2 + \cdots\right),$$

whence follows

$$e^{\alpha x}(e^x-1)^M = x^M + \left(\frac{M}{2} + \alpha\right) x^{M+1} + \cdots \tag{C. 54}$$

The lemma readily follows by equating the coefficients of x^N in (C.52) and (C.54). Q.E.D.

It should be noted that for $\alpha = 0$ the above identity (C.51) reduces to an important result closely related to the definition of STIRLING numbers of the second kind (e.g. see ABRAMOWITZ and STEGUN [1, p. 824] or JORDAN [9, pp. 168-169]; see also SCHWATT [13, pp. 100-101]).

The above lemma allows us to easily prove the following important result, whose proof was generously provided to the author by G. E. LATTA.

LEMMA C.2: For any integers M and N such that $0 \leq N \leq M$ and real numbers α and $\beta \geq 0$, we have that

$$\sum_{k=0}^{M} \binom{M}{k} \frac{(-1)^k (k+\alpha)^N}{\{1 + \beta(k+\alpha)\}} = \frac{(-1)^N M!\, \beta^{M-N}}{\prod_{k=0}^{M} \{1 + \beta(k+\alpha)\}} . \tag{C.55}$$

PROOF (LATTA [10]). Define $F(\beta, M, N)$ as follows

$$F(\beta, M, N) = \sum_{k=0}^{M} \binom{M}{k} \frac{(-1)^k (k+\alpha)^N}{\{1 + \beta(k+\alpha)\}} . \tag{C.56}$$

Consider now

$$\beta F(\beta, M, N) = \sum_{k=0}^{M} \binom{M}{k} \frac{(-1)^k (k+\alpha)^{N-1}}{\{1 + \beta(k+\alpha)\}} \{[\beta(k+\alpha) + 1] - 1\},$$

whence

$$\beta F(\beta, M, N) = F(0, M, N-1) - F(\beta, M, N-1) . \tag{C.57}$$

By Lemma C.1, however, we know that $F(0, M, L) = 0$ for all integers L such that $0 \leq L < M$, whence $F(0, M, N-1) = 0$ for all integers N such

that $1 \leq N \leq M$. Hence, (C.57) yields that for $0 < N \leq M$

$$F(\beta, M, N) = -\frac{1}{\beta} F(\beta, M, N-1) ,$$

whence for $0 \leq N \leq M$

$$F(\beta, M, N) = (-1)^N \beta^{-N} F(\beta, M, 0) . \tag{C.58}$$

Recalling that the gamma function satisfies [6; 12]

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

for all positive real numbers x and y (and also serves to define the beta function), we consider

$$F(\beta, M, 0) = \frac{1}{\beta} \sum_{k=0}^{M} \binom{M}{k} (-1)^k \frac{1}{(k + \alpha + 1/\beta)} ,$$

which may also be written as

$$F(\beta, M, 0) = \frac{1}{\beta} \sum_{k=0}^{M} \binom{M}{k} (-1)^k \int_0^1 t^{k+\alpha+1/\beta-1} dt ,$$

whence follows

$$F(\beta, M, 0) = \frac{1}{\beta} \int_0^1 t^{\alpha+1/\beta-1} (1-t)^{(M+1)-1} dt ,$$

and finally

$$F(\beta, M, 0) = \frac{1}{\beta} \frac{\Gamma(\alpha + 1/\beta)\, M!}{\Gamma(M + \alpha + 1/\beta + 1)} . \tag{C.59}$$

Observing that for $M \geq 0$

$$\frac{\Gamma(\alpha + 1/\beta)}{\Gamma(M + \alpha + 1/\beta + 1)} = \frac{1}{\prod_{j=0}^{M} (\alpha + 1/\beta + M - j)} ,$$

one may also readily show that for $M \geq 0$

$$\frac{\Gamma(\alpha + 1/\beta)}{\Gamma(M + \alpha + 1/\beta + 1)} = \frac{\beta^{M+1}}{\prod_{k=0}^{M} \{1 + \beta(k + \alpha)\}} . \tag{C.60}$$

Combining (C.58), (C.59), and (C.60), we find that for $0 \leq N \leq M$ and α and $\beta \geq 0$.

$$F(\beta, M, N) = \frac{(-1)^N M! \, \beta^{M-N}}{\prod_{k=0}^{M} \{1 + \beta(k + \alpha)\}} ,$$

whence follows the lemma. Q.E.D.

REFERENCES FOR APPENDIX C

1. M. Abramowitz and I. A. Stegun (Editors), Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series, No. 55, Washington, D.C., 1964.

2. P. M. Batchelder, An Introduction to Linear Difference Equations, Harvard University Press, Cambridge, Massachusetts, 1927 (reprinted by Dover Publications, Inc., New York, 1967).

3. R. H. Brown, "The Solution of a Certain Difference Equation with Applications to Probability, Ph.D. Thesis, Columbia University, New York, New York, 1951 (also available from University Microfilms International, P. O. Box 1764, Ann Arbor, Michigan 48106 as Publication No. 3325).

4. R. H. Brown, "Theory of Combat: The Probability of Winning," Opns. Res. 11, 418-425 (1963).

5. G. M. Clark, "The Combat Analysis Model," Ph.D. Thesis, The Ohio State University, Columbus, Ohio, 1969 (also available from University Microfilms International as Publication No. 69-15,905).

6. E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, London, 1935.

7. R. Courant and F. John, Introduction to Calculus and Analysis, Volume One, Interscience, New York, 1965.

8. F. B. Hildebrand, Finite-Difference Equations and Simulations, Prentice-Hall, Englewood Cliffs, New Jersey, 1968.

9. C. Jordan, Calculus of Finite Differences, Second Edition, Chelsea Publishing Co., New York, 1950.

10. G. E. Latta, private communication, March 1980.

11. K. S. Miller, An Introduction to the Calculus of Finite Differences and Difference Equations, Henry Holt and Co., New York, 1960.

12. E. D. Rainville, Special Functions, The Macmillan Co., New York, 1960 (reprinted by Chelsea Publishing Co., Bronx, New York, 1971).

13. I. J. Schwatt, An Introduction to the Operations with Series, The University of Pennsylvania Press, Philadelphia, 1924 (reprinted by Chelsea Publishing Co., New York, 1962).